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M.I.N Institute of Media, Information, and Network

Chapter 4 The Continue-Time Fourier Transform

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Foreword of the Chapter

- By exploiting the properties of superposition and time invariance, if we know the response of an LTI system to some inputs, we actually know the response to many inputs

If $$x_k[n] \rightarrow y_k[n]$$

Then $$\sum_k a_k x_k[n] \rightarrow \sum_k a_k y_k[n]$$

- If we can find sets of "basic" signals so that
  - We can represent rich classes of signals as linear combinations of these building block signals.
  - The response of LTI Systems to these basic signals are both simple and insightful.

- Candidate sets of basic signal
  - Unit impulse function and its delays

$$\delta(t) / \delta[n]$$

  - Complex exponential signals (Eigenfunctions of all LTI systems) $e^{j\omega t} / e^{st}$ $e^{j\Omega n} / z^n$
- In this Chapter, we will focus on: why, how, what
  - Can we represent aperiodic signals as "sums or integrals" of complex exponentials
  - How to represent aperiodic signals as "sums or integrals" of complex exponentials
  - What kinds of aperiodic signals can we represent as "sums or integrals" of complex exponentials? (how large types of such signals can benefit from the Fourier Transform?)

Topic

- 4.0 Introduction
- 4.1 The Continuous-Time Fourier Transform
- 4.2 The Fourier Transform for Periodic Signals
- 4.3 Properties of the Continuous-Time Fourier Transform
- 4.4 The Convolution Property
- 4.5 The multiplication Property
- 4.6 System Characterized by Linear Constant-Coefficient Differential Equations

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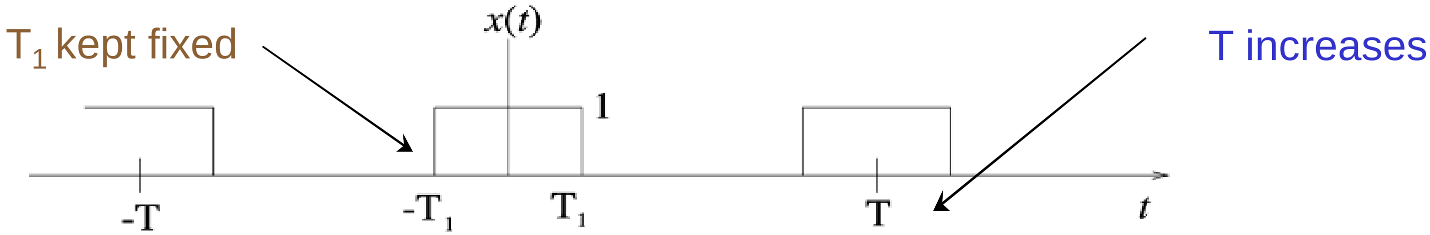
4.0 Introduction

- Fourier Series Representation
  - It decomposes any periodic function or periodic signal into the sum of a (possibly infinite) set of simple oscillating functions, namely sines and cosines (or, equivalently, complex exponentials). The discrete-time Fourier transform is a periodic
- Fourier Transform
  - A representation of aperiodic signals as linear combinations of complex exponentials

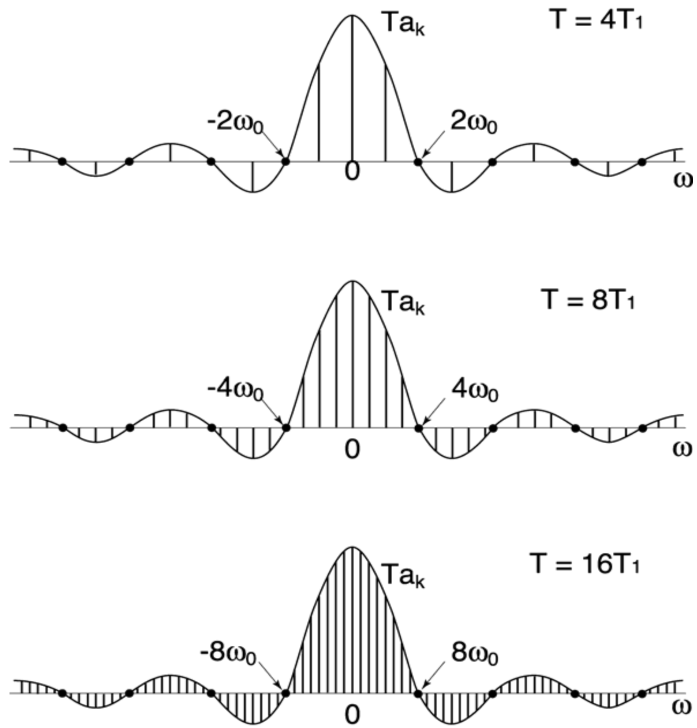
Motivating Example



$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

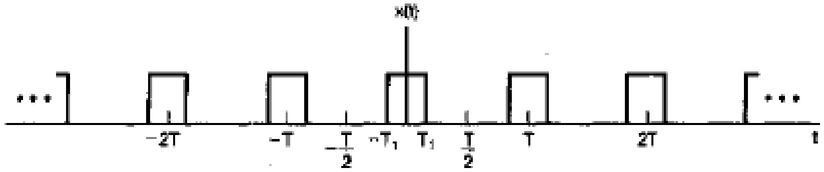
$$\Downarrow$$

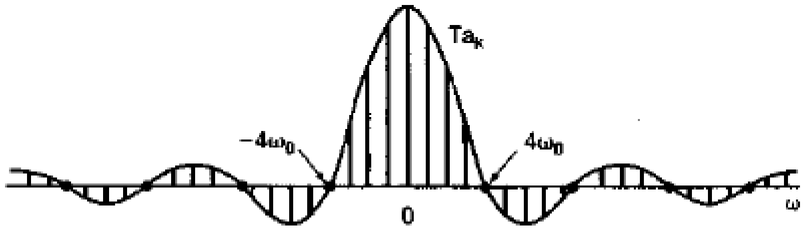
$$T a_k = \left.\frac{2\sin \omega T_1}{\omega}\right|_{\omega = k\omega_0}$$



Discrete frequency points become denser in ω as T increases

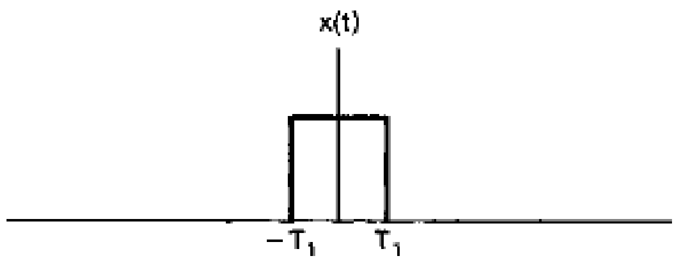
- Then for periodic square wave, the spectrum of x(t), i.e. $\{a_k\}$, are $a_k = \dfrac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$, the spectrum space is $\omega_0 = \dfrac{2\pi}{T}$





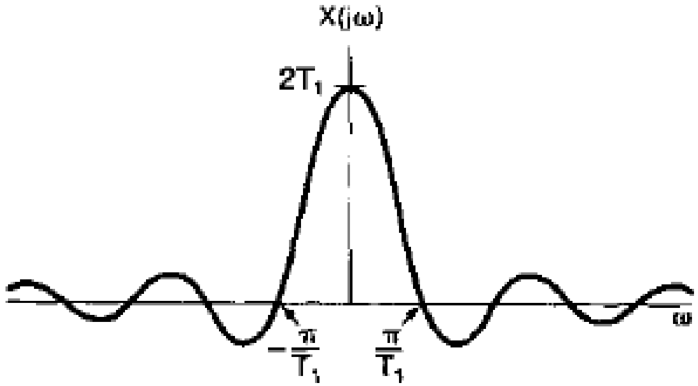
(b)

- *Then for square pulse, the spectrum X(jω) are* $\dfrac{2\sin(\omega T_1)}{\omega}$ *, the spectrum space is* $\omega_0 = \dfrac{2\pi}{T} \to 0$ *, i.e. the complex exponentials occur at a continuum of frequencies*



(a)

$\leftrightarrow$



(b)

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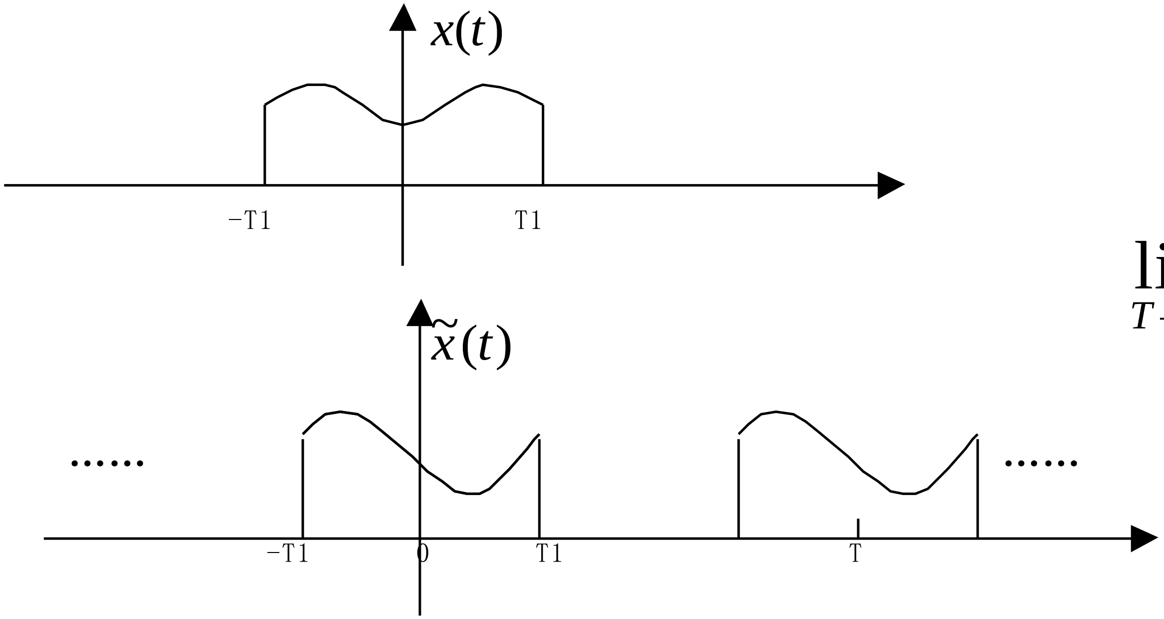
4.1.1 Development

- **To derive the spectrum for aperiodic signals *x(t)*, we can approximate it by a periodic signal $\tilde{x}(t)$ with infinite period *T***



$$\lim_{T\to\infty}\tilde{x}(t) = x(t)$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \left(\omega_0 = \frac{2\pi}{T}\right)$$

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}dt$$

$\uparrow$

$\tilde{x}(t) = x(t)$ in this interva

$$= \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt \qquad (1)$$

Assuming (1) is converged, we define

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

then Eq.(1) $\Rightarrow$

$$a_k = \frac{X(jk\omega_0)}{T}$$

- Thus

$$\tilde{x}(t) = \sum_k a_k e^{jk\omega_0 t} = \sum_k \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0$$

- When $T \rightarrow \infty$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$	Synthesis equation
$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$	Analysis equation

4.1.2 Convergence

- What kinds of signals can be represented in Fourier Transform (satisfies one of the following 2 conditions)
  - 1、Finite energy

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

Then we are guaranteed that:

- $X(j\omega)$ is finite
- $\int_{-\infty}^{\infty} |e(t)|^2 dt = 0$

$(e(t) = \hat{x}(t) - x(t) \qquad \hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega)$

- 2、Dirichlet conditions, require that
  - $x(t)$ be absolutely integrable
  - $x(t)$ have a finite number of maxima and minima within any finite interval
  - $x(t)$ have a finite number of discontinuities within any finite interval. Furthermore, each of these discontinuities must be finite

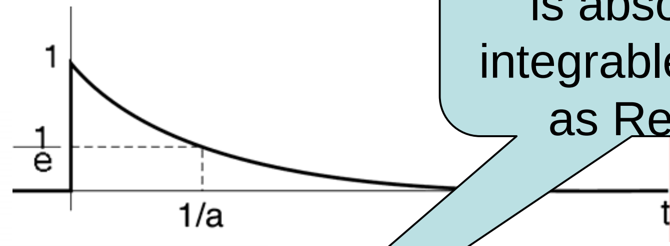
Then we guarantee that

- $\hat{x}(t)$ is equal to $x(t)$ for any t except at a discontinuity, where it is equal to the average of the values on either side of the discontinuity
- $X(j\omega)$ is finite

Examples

- Exponential function

$$x(t) = e^{-at}u(t), a > 0$$



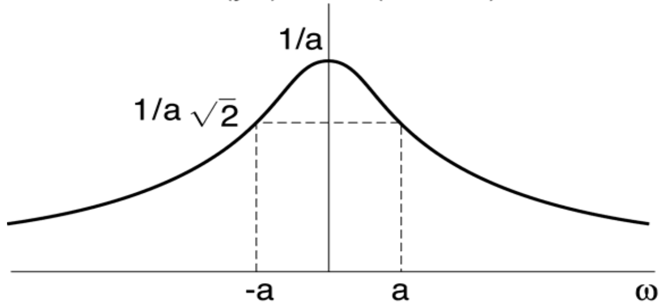
If α is complex, x(t) is absolutelty integrable as long as Re{α}>0

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_0^{\infty} \underbrace{e^{-at}e^{-j\omega t}}_{e^{-(a+j\omega)t}} dt$$

$$= -\left(\frac{1}{a+j\omega}\right)e^{-(a+j\omega)t}\Big|_0^{\infty} = \frac{1}{a+j\omega}$$

Magnitude Spectrum

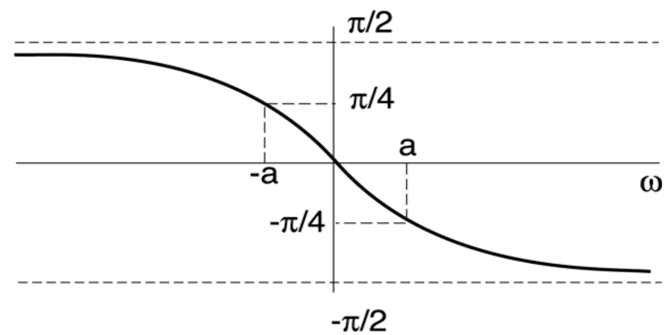
$|X(j\omega)| = 1/(a^2+\omega^2)^{1/2}$



Even symmetry

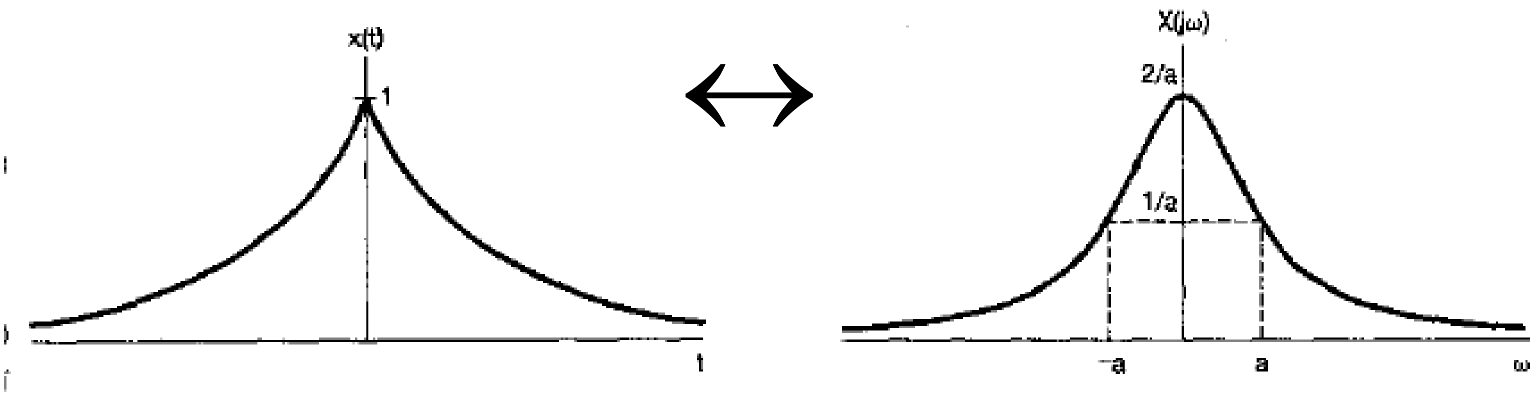
Phase Spectrum

$\angle X(j\omega) = -\tan^{-1}(\omega/a)$



Odd symmetry

$$x(t) = e^{-\alpha|t|},\ \alpha > 0 \leftrightarrow X(j\omega) = \frac{2\alpha}{\alpha^2 + \omega^2}$$



Examples

- Unit impulse

$$x(t) = \delta(t)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t} dt = 1$$

- DC Signal

$$x(t) = 1 \leftrightarrow X(j\omega) = 2\pi\delta(\omega)$$

$$\because \frac{1}{2\pi}\int \delta(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}$$

$$\therefore \frac{1}{2\pi} \leftrightarrow \delta(\omega)$$

$$\therefore 1 \leftrightarrow 2\pi\delta(\omega)$$

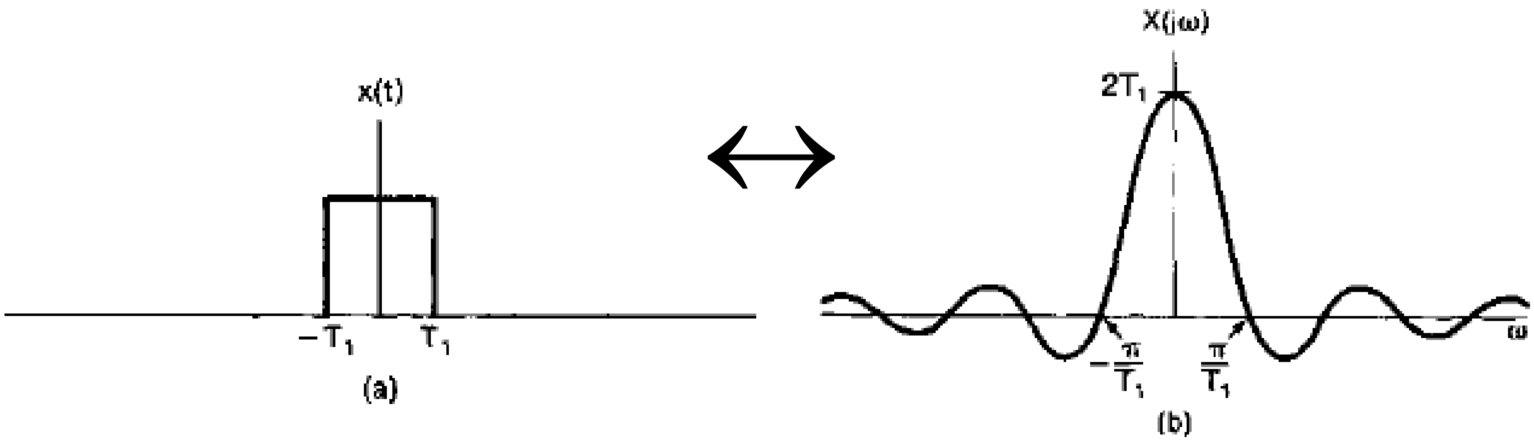
Example

$$S_a(x) = \frac{sinx}{x}$$

$$sinx(x) = \frac{sin\pi x}{\pi x}$$

- Rectangle Pulse Signal

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \leftrightarrow X(j\omega) = 2\frac{\sin \omega T_1}{\omega} = 2T_1 S_a(\omega T_1)$$



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- For a periodic signal x(t) with fundamental frequency $\omega_0 = \frac{2\pi}{T}$, what's its FT?

$$\because x(t) = \sum_k a_k e^{jk\omega_0 t}$$

$$\therefore \Im[x(t)] = \Im[\sum_k a_k e^{jk\omega_0 t}] = \sum_k a_k \Im[e^{jk\omega_0 t}]$$

the question becomes: $e^{jk\omega_0 t} \leftrightarrow ?$

- Thanks to the impulse function, suppose

$$X(j\omega) = \delta(\omega - \omega_0)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega - \omega_0)e^{j\omega t}d\omega = \frac{1}{2\pi}e^{j\omega_0 t}$$

- That is $e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$

— All the energy is concentrated in one frequency — ω_{o},

- So

$$x(t) = \sum_k a_k e^{jk\omega_0 t} \leftrightarrow X(j\omega) = \sum_k 2\pi a_k\,\delta(\omega - k\omega_0)$$

- So for a periodic signal x(t) with fundamental frequency $\omega_0 = \frac{2\pi}{T}$, its FT is:
  - Fourier Series Coefficient

$$x(t) = \sum a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}dt$$

  - Fourier Transform

$$x(t) \leftrightarrow X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0), \quad \omega_0 = \frac{2\pi}{T}$$

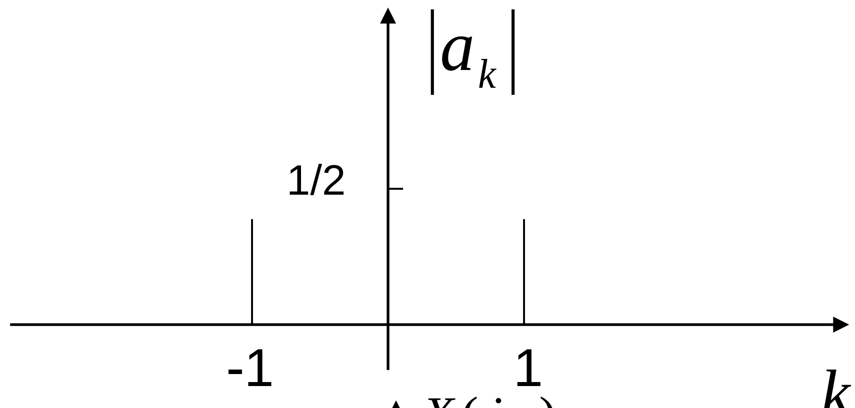
- The FT can be interpreted as a train of impulses occurring at the harmonically related frequencies and for which the area of the impulse at the k^{th} harmonic frequency $k\omega_o$ is 2π times the kth F.S. coefficient a_k

- Example: $x(t) = \cos(\omega_0 t)$

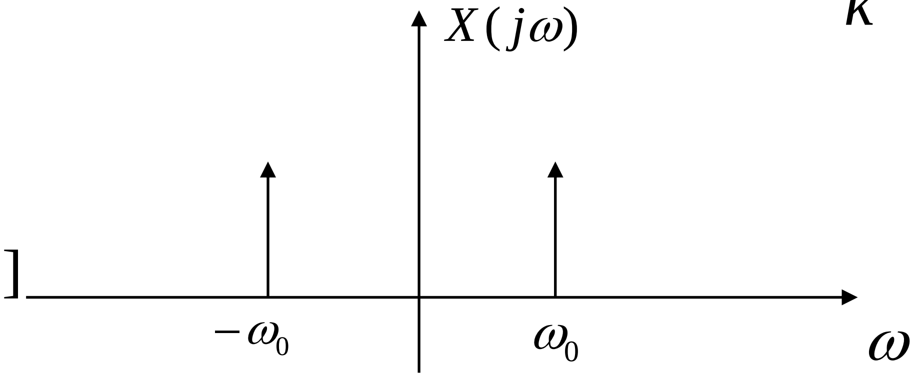
$$x(t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

$$\therefore\ a_1 = a_{-1} = \frac{1}{2}$$

$$a_k = 0,\quad k \neq \pm 1$$



$$X(j\omega) = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$



Similarly:

$$\sin\omega_0 t \leftrightarrow j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$$

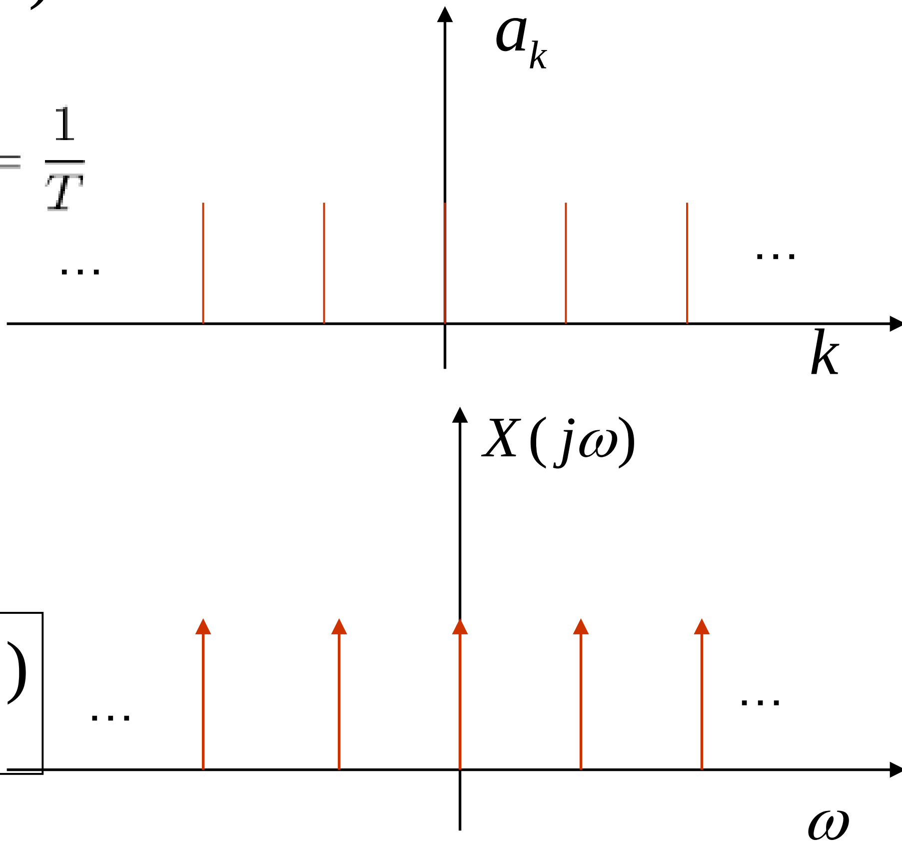
- Example: $x(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT)$

$$x(t)\leftrightarrow a_k=\frac{1}{T}\int_{-T/2}^{T/2}x(t)e^{-jk\omega_0 t}dt=\frac{1}{T}$$

$$\Downarrow$$

$$X(j\omega)=\sum_{n=-\infty}^{\infty}\underbrace{\frac{2\pi}{T}}_{2\pi a_k}\delta(\omega-\underbrace{\frac{k2\pi}{T}}_{k\omega_0})$$

$$\boxed{\sum_k\delta(t-kT)\leftrightarrow\omega_0\sum_k\delta(\omega-k\omega_0)}$$



Same function in the frequency-domain!

Note: (period in *t*) *T* ⇔ (period in ω) 2π/*T*

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- Linearity

$$x(t) \leftrightarrow X(j\omega) \qquad y(t) \leftrightarrow Y(j\omega)$$

$$\boxed{ax(t) + by(t) \leftrightarrow aX(j\omega) + bY(j\omega)}$$

- Time Shifting

$$x(t) \leftrightarrow X(j\omega)$$

$$\boxed{x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(j\omega)}$$

- Time and Frequency Scaling

$$x(t) \leftrightarrow X(j\omega)$$

$$\boxed{x(at) \leftrightarrow \frac{1}{|a|} X(\frac{j\omega}{a})}$$

for $a = -1$ $\quad x(-t) \leftrightarrow X(-j\omega)$

compressed in time ⇔ stretched in frequency

$$x(at+b) \leftrightarrow ?$$

$$\boxed{x(at+b) \leftrightarrow \frac{1}{|a|} X(\frac{j\omega}{a}) e^{j\frac{\omega b}{a}}}$$

- Example: Determine the Fourier Transform of the following signals

1. $x(t) = e^{-2t}u(t)$

2. $x(t) = e^{-2(t-1)}u(t)$

3. $x(t) = e^{-2t}u(t-1)$

- Differentiation

$$x(t) \leftrightarrow X(j\omega)$$

$$\boxed{\frac{dx(t)}{dt} \leftrightarrow j\omega X(j\omega)}$$

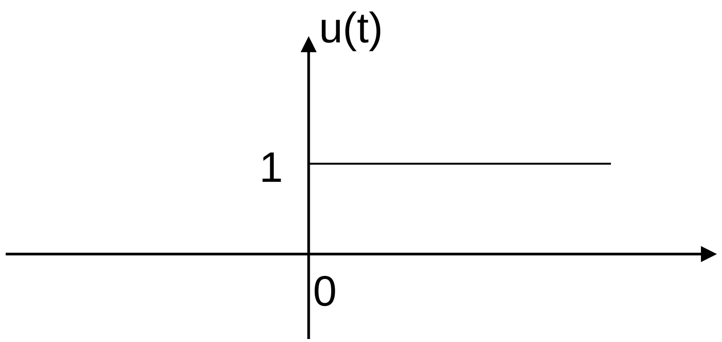
- The differentiation operation enhances high-frequency components in the effective frequency band of a signal
- Without any further information about the DC component of the original signal, we cannot completely recover it from its differentials

- Integration

$$g(t) = \int_{-\infty}^{t} x(\tau)d\tau \qquad x(t) \leftrightarrow X(j\omega)$$

$$\boxed{g(t) = x(t) * u(t) \leftrightarrow G(j\omega) = X(j\omega)U(j\omega)}$$

where u(t) is the unit step function, defined as

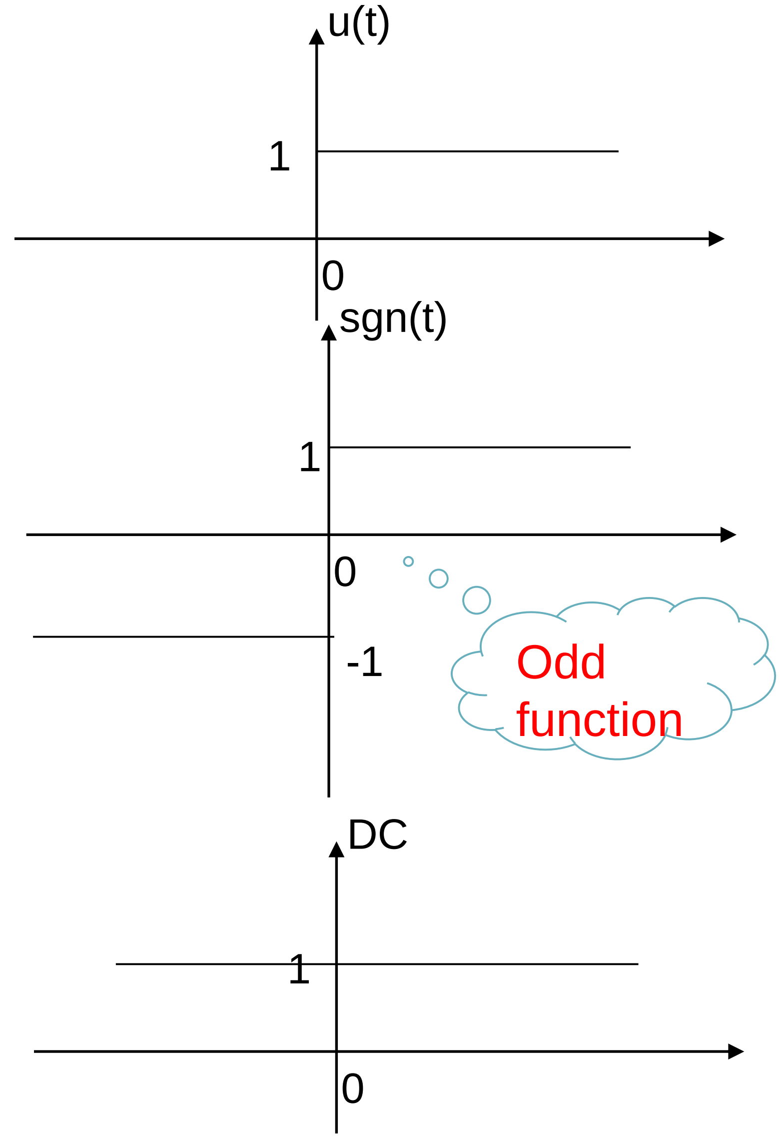


- $u(t) = \frac{1+sgn(t)}{2}$

$$x(t) = 1 \leftrightarrow X(j\omega) = 2\pi\delta(\omega)$$

- $x(t) = sgn(t) \leftrightarrow X(j\omega) = \frac{2}{j\omega}$

$$\boxed{U(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)}$$



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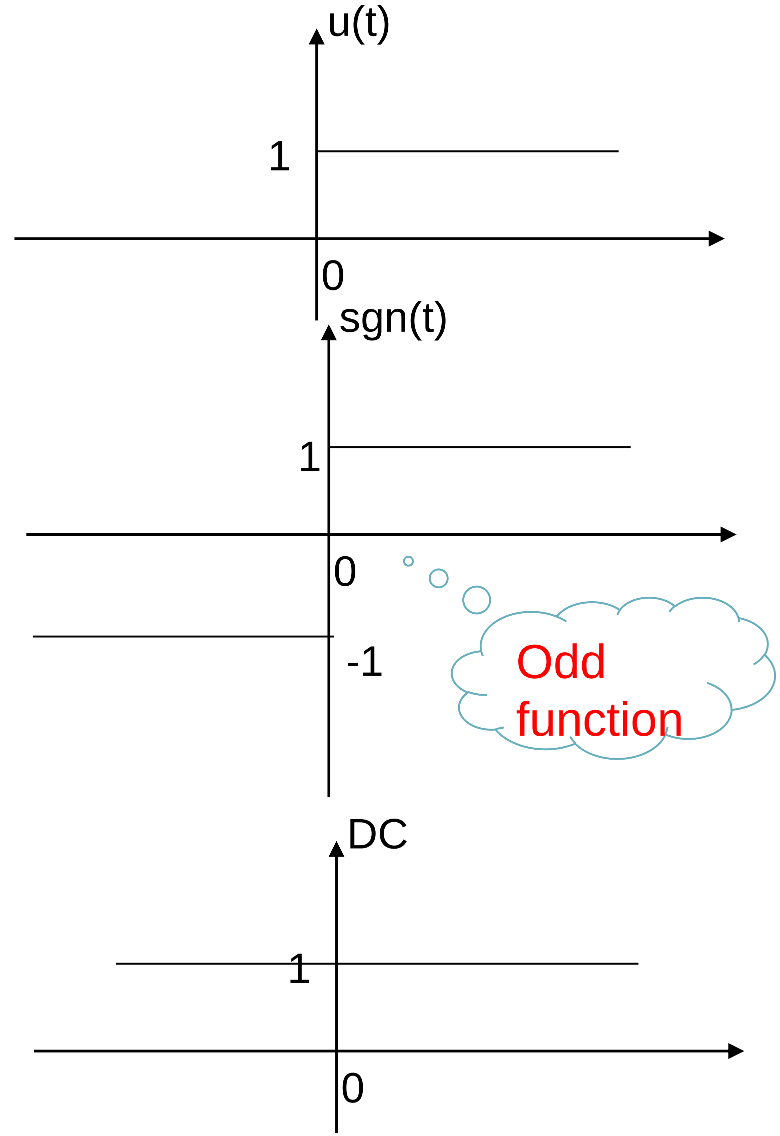


$$U(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

according to:

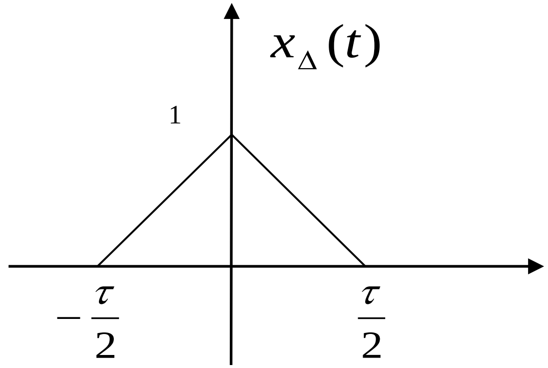
$$g(t) = x(t) * u(t) \leftrightarrow G(j\omega) = X(j\omega)U(j\omega)$$

$$g(t) = \int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow G(j\omega) = \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

The integration operation diminishes high-frequency components in the effective frequency band of a signal

- Example: triangle pulse

$$x_{\Delta}(t)=\begin{cases}1-\frac{2|t|}{\tau}, & |t|\leq\frac{\tau}{2}\\ 0, & |t|>\frac{\tau}{2})\end{cases}$$

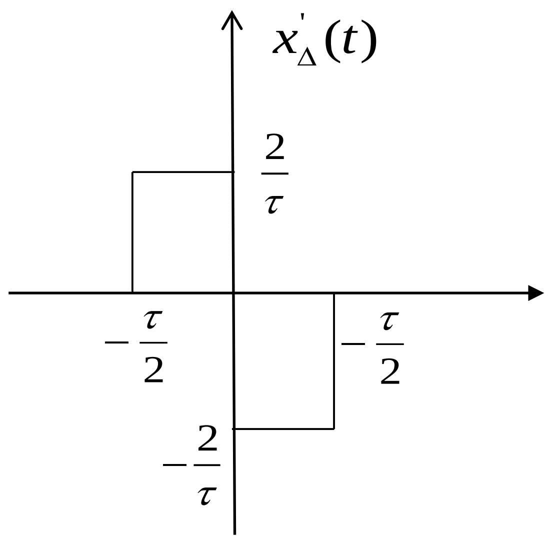


$$x_{\Delta}^{'}(t)\leftrightarrow X_1(j\omega)$$

$$X(j\omega)=\frac{X_1(j\omega)}{j\omega}+\pi X_1(0)\delta(\omega)$$

Since $X_1(0)=\int x_{\Delta}^{'}(t)dt=0$

$$X(j\omega)=\frac{8\sin^2(\frac{\omega\tau}{4})}{\omega^2\tau}=\frac{\tau}{2}Sa^2(\frac{\omega\tau}{4})$$



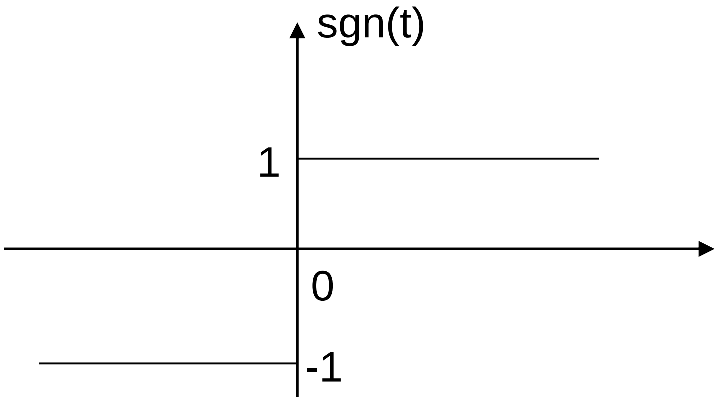
- 2 approaches to calculate X(0)：

1. $X(0) = X(j\omega)\big|_{\omega=0}$

2. $X(0) = \int_{-\infty}^{\infty} x(t)dt$

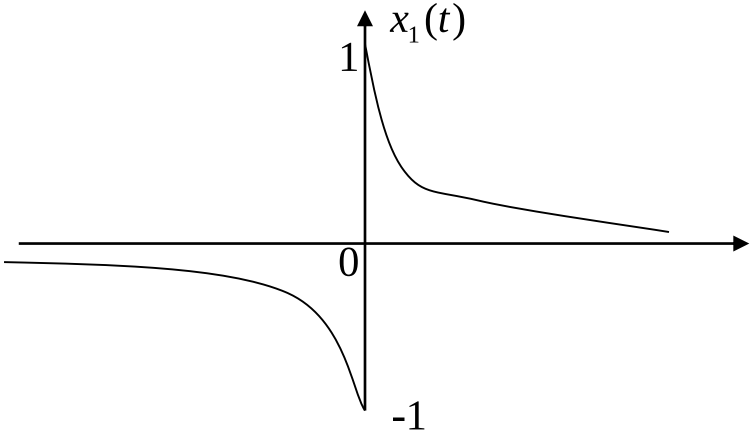
- Example: sgn(t)

$$x(t) = \operatorname{sgn}(t) = \begin{cases} 1, & (t > 0) \\ -1, & (t < 0) \end{cases}$$



- By defining the sgn function as a special exponential function

$$x_1(t)=\begin{cases} e^{-\alpha t}, & (t>0) \\ -e^{\alpha t}, & (t<0) \end{cases} \quad \alpha>0$$



$$\operatorname{sgn}(t)=\lim_{\alpha\to 0} x_1(t)$$

$$\therefore \operatorname{sgn}(t)\leftrightarrow \lim_{\alpha\to 0} X_1(j\omega)=\frac{2}{j\omega}$$

- By representing the sgn function in terms of unit step functions

$$\operatorname{sgn}(t)=u(t)-u(-t)$$

$$\therefore \operatorname{sgn}(t)\leftrightarrow (\pi\delta(\omega)+\frac{1}{j\omega})-[\pi\delta(-\omega)+\frac{1}{-j\omega}]=\frac{2}{j\omega}$$

- By exploiting integration property

$$\because \operatorname{sgn}(t) = 2u(t) - 1 \quad \therefore \operatorname{sgn}'(t) = 2\delta(t) = x_1(t)$$

Suppose $x_1(t) = \dfrac{dx(t)}{dt} \leftrightarrow X_1(j\omega)$

When x(-∞)≠0

$$\because \int_{-\infty}^{t} x_1(t)dt = x(t) - x(-\infty)$$

$$\therefore x(t) = \int_{-\infty}^{t} x_1(t)dt + x(-\infty)$$

$$\therefore X(j\omega) = \frac{X_1(j\omega)}{j\omega} + \pi X_1(0)\delta(\omega) + 2\pi x(-\infty)\delta(\omega)$$

$$\because X_1(j\omega) = 2 \qquad \therefore \operatorname{sgn}(t) = \frac{2}{j\omega} + \pi \cdot 2\delta(\omega) + 2\pi \cdot (-1) \cdot \delta(\omega) = \frac{2}{j\omega}$$

- Duality
  - Both time and frequency are continuous and in general aperiodic

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Same except for these differences

$$\Downarrow$$

  - Suppose f() and g() are two functions related by

$$f(r) = \int_{-\infty}^{\infty} g(\tau)e^{-jr\tau}d\tau$$

Let $\tau = t$ and $r = \omega$: $\quad x_1(t) = g(t) \longleftrightarrow X_1(j\omega) = f(\omega)$

Let $\tau = -\omega$ and $r = t$: $\quad x_2(t) = f(t) \longleftrightarrow X_2(j\omega) = 2\pi g(-\omega)$

$$x(t) \leftrightarrow X(j\omega) \qquad \boxed{X(t) \leftrightarrow 2\pi x(-j\omega)}$$

- Example

$$x(t)=\frac{\sin Wt}{\pi t} \leftrightarrow X(j\omega)=\begin{cases}1, & |\omega|<W \\ 0, & |\omega|>W\end{cases}$$

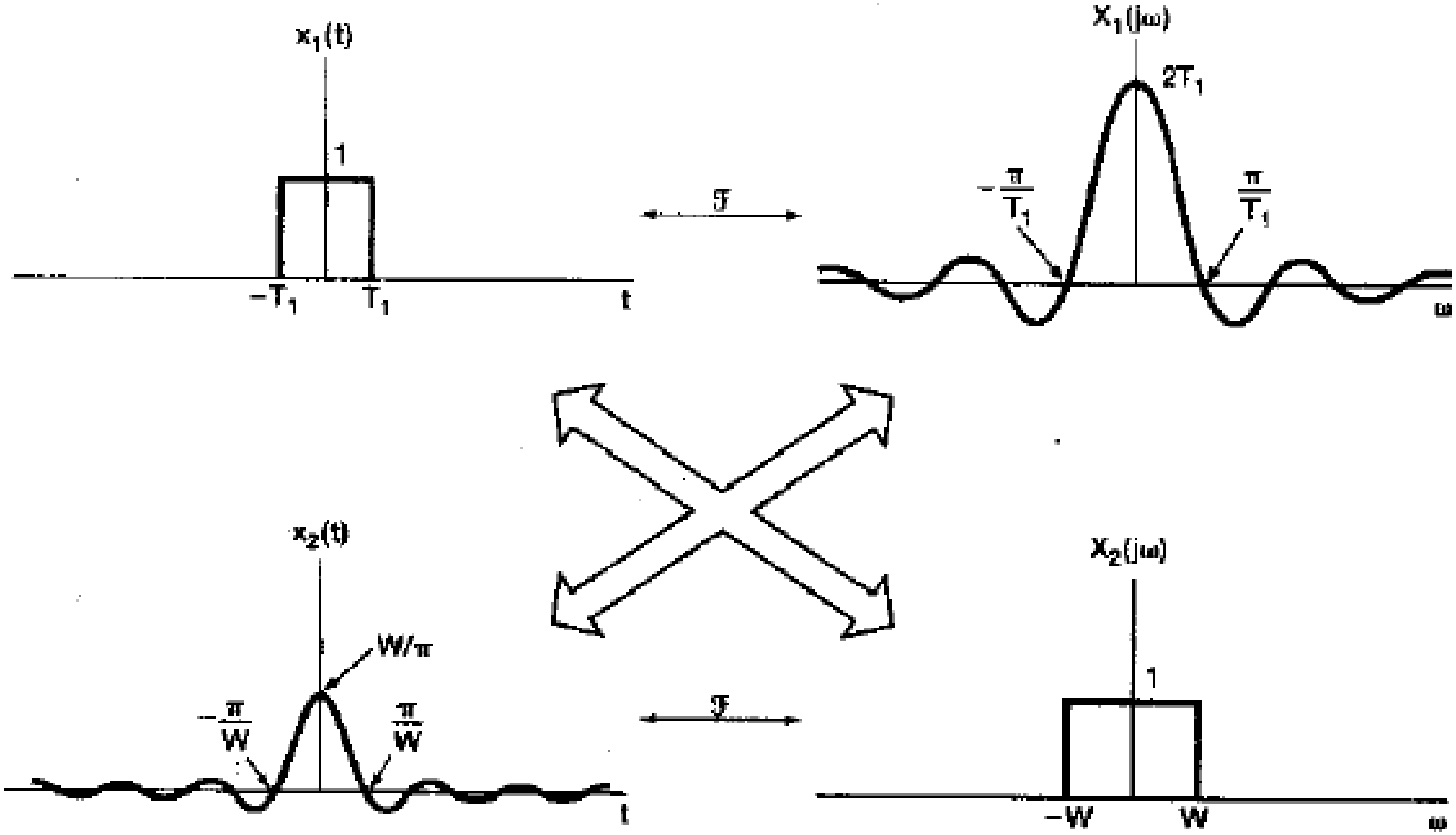


图 4.17 (4.36)式和(4.37)式两对傅里叶变换之间的关系

- Example $x(t) = \dfrac{1}{\pi t}$

$$x(t) = \operatorname{sgn}(t) \quad \leftrightarrow \quad X(j\omega) = \begin{cases} 2/j\omega & \omega \neq 0 \\ 0 & \omega = 0 \end{cases}$$

$$X(j\omega) = -j\operatorname{sgn}(\omega)$$

- Example $x(t) = \dfrac{1}{1+t^2}$

$$x(t) = e^{-\alpha|t|}, \operatorname{Re}\{\alpha\} > 0 \quad \leftrightarrow \quad X(j\omega) = \frac{2\alpha}{|\alpha|^2 + \omega^2}$$

$$X(j\omega) = \pi e^{-|\omega|}$$

- Other duality properties
  - （1） Frequency Shifting

$$x(t) \leftrightarrow X(j\omega)$$

$$\boxed{e^{j\omega_0 t}x(t) \leftrightarrow X(j(\omega-\omega_0))}$$

- Example:

$$\cos\omega_0 t = \frac{1}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}]$$

$$\leftrightarrow \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$$

$$\sin\omega_0 t = \frac{1}{2j}[e^{j\omega_0 t} - e^{-j\omega_0 t}]$$

$$\leftrightarrow \frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$$

$$= j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$$

- （2）Differentiation in frequency domain

$$-jtx(t) \leftrightarrow \frac{dX(j\omega)}{d\omega}$$

$$tx(t) \leftrightarrow j\frac{dX(j\omega)}{d\omega}$$

- （3）Integration in frequency domain

$$-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \leftrightarrow \int_{-\infty}^{\omega} X(\lambda)d\lambda$$

when x(0)=0,

$$\frac{x(t)}{t} \leftrightarrow -j\int_{-\infty}^{t} X(\lambda)d\lambda$$

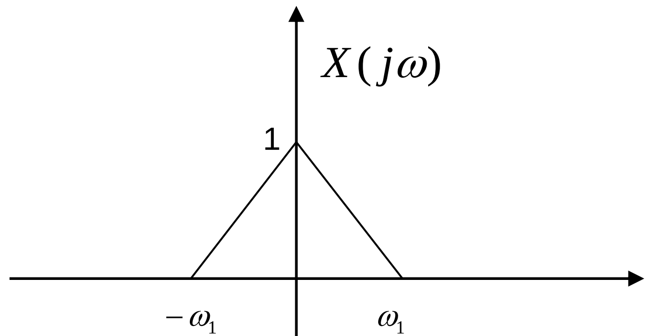
- Example: $x(t) = te^{-2t}u(t) \leftrightarrow ?$

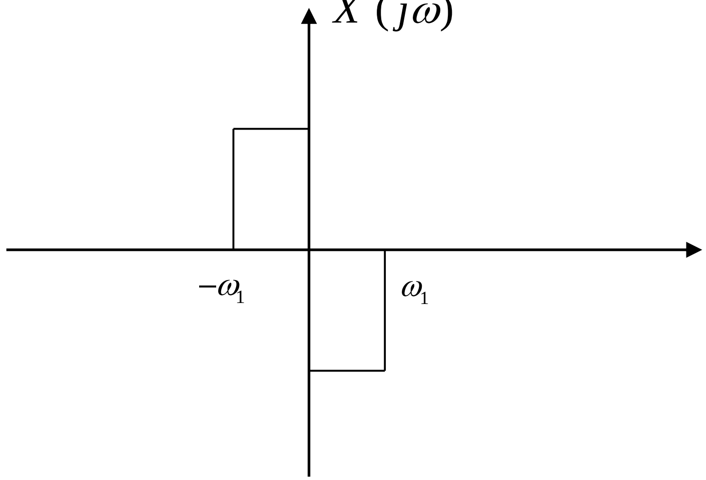
$$\because e^{-2t}u(t) \leftrightarrow \frac{1}{2 + j\omega}$$

$$\therefore te^{-2t}u(t) \leftrightarrow j\frac{d}{d\omega}\left(\frac{1}{2 + j\omega}\right) = \frac{1}{(2 + j\omega)^2}$$

$$x(t) = te^{-2t}u(t-1) \leftrightarrow ?$$

- Example :To determine x(t) according to X(jω)





Hints: To exploit the differentiation property in frequency domain

$$X'(j\omega) \leftrightarrow x_1(t)$$

$$x(t) = \frac{x_1(t)}{-jt} + \pi x_1(0)\delta(t)$$

又 $x_1(0)=\int X'(j\omega)d\omega = 0$

- Conjugation and Conjugate Symmetry

$$x(t) \leftrightarrow X(j\omega)$$

$$\boxed{x^*(t) \leftrightarrow X^*(-j\omega)}$$

If x(t) is real valued

$$X(j\omega) = X^*(-j\omega)$$ —Conjugate Symmetry

① $$X(j\omega) = \mathrm{Re}[X(j\omega)] + jI_m[X(j\omega)]$$

$$\mathrm{Re}[X(j\omega)] = \mathrm{Re}[X(-j\omega)]$$

$$\mathrm{Im}[X(j\omega)] = -\mathrm{Im}[X(-j\omega)]$$

② $$X(j\omega) = |X(j\omega)|\, e^{j\angle X(j\omega)}$$

$$|X(j\omega)| = |X(-j\omega)|$$

$$\angle X(j\omega) = -\angle X(-j\omega)$$

③ $x(t)$ *real and even* $\leftrightarrow X(j\omega)$ *real and even*

$x(t)$ *real and odd* $\leftrightarrow X(j\omega)$ *purely imaginary and odd*

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] \leftrightarrow Re[X(j\omega)]$$

$$x_0(t) = \frac{1}{2}[x(t) - x(-t)] \leftrightarrow jI_m[X(j\omega)]$$

- Example:

$$x(t) = e^{-\alpha|t|} = e^{-\alpha t}u(t) + e^{\alpha t}u(-t) = 2E_v\{e^{-\alpha t}u(t)\}$$

$$\because e^{-\alpha t}u(t) \leftrightarrow \frac{1}{\alpha + j\omega}$$

$$\therefore x(t) \leftrightarrow 2\,\mathrm{Re}\{\frac{1}{2 + j\omega}\} = \frac{2\alpha}{\alpha^2 + \omega^2}\bigg|_{\alpha=2}$$

- Parseval's Relation

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = \int_{-\infty}^{\infty} |X(f)|^2 df$$

$|X(f)|^2$ ——Energy per unit frequency (Hz)

$|X(j\omega)|^2$ ——Energy-density Spectrum

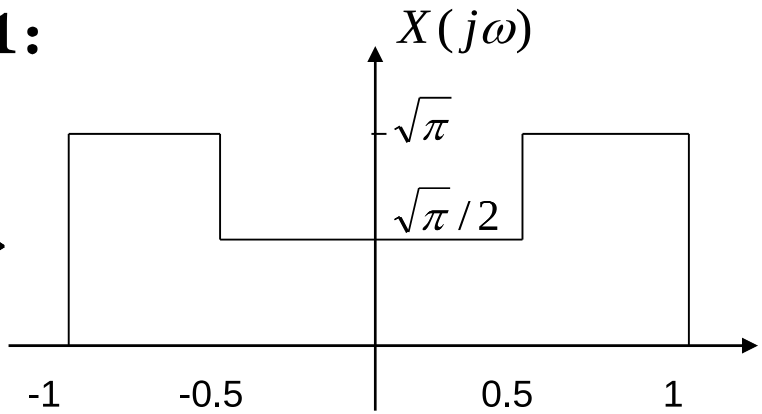
and:

$$\lim_{T\to\infty} \frac{1}{T} \int_{T} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{|X(j\omega)|^2}{T} d\omega$$

$\lim_{T\to\infty} \frac{|X(j\omega)|^2}{T}$ ——Power-density Spectrum

- Example1:

$$x(t) \leftrightarrow$$



To determine $\int |x(t)|^2 \, dt$

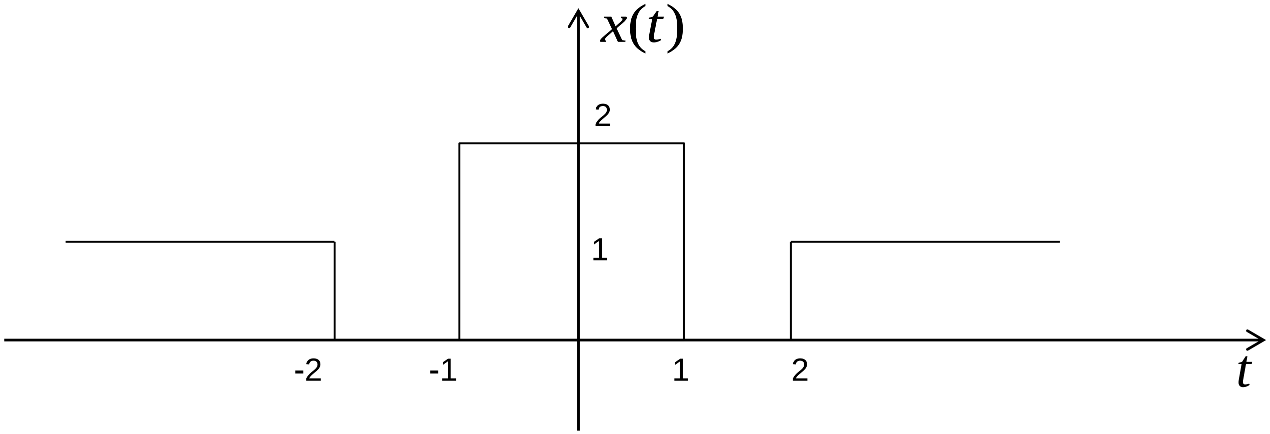
- Example2:

$$x(t) = \frac{\sin 2t}{\pi t}$$

To determine $\int |x(t)|^2 \, dt$

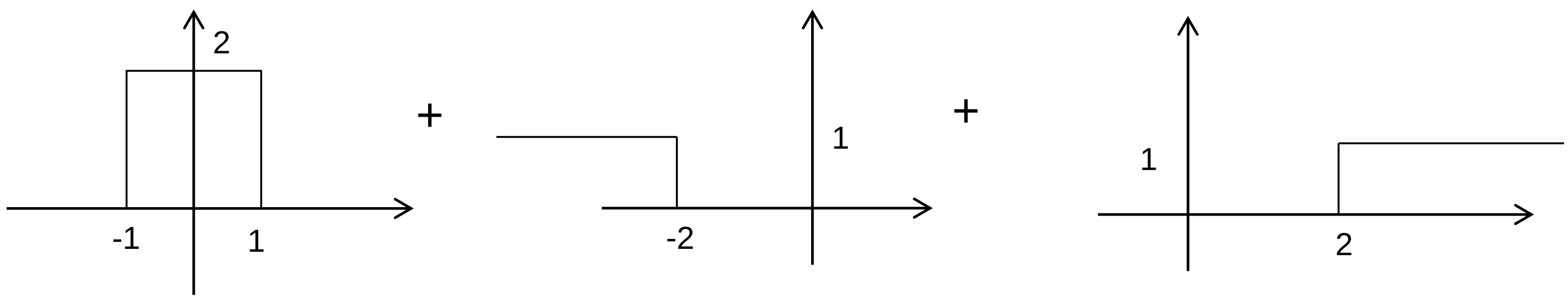
- Example:

To use the FT of typical signals and FT properties to determine the FT of the following signals

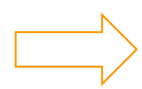


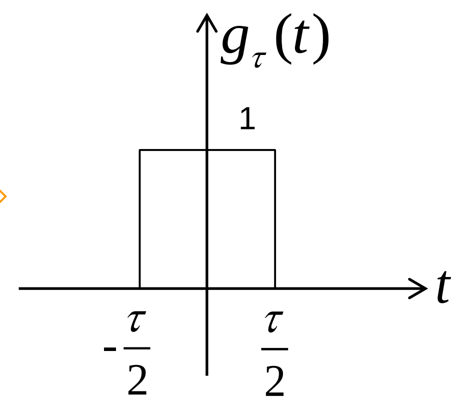
- **Solution 1:**

$$x(t) = 2g_2(t) + u(-t-2) + u(t-2)$$



$g_\tau(t)$ is the rectangle pulse with width of τ and unit magnitude



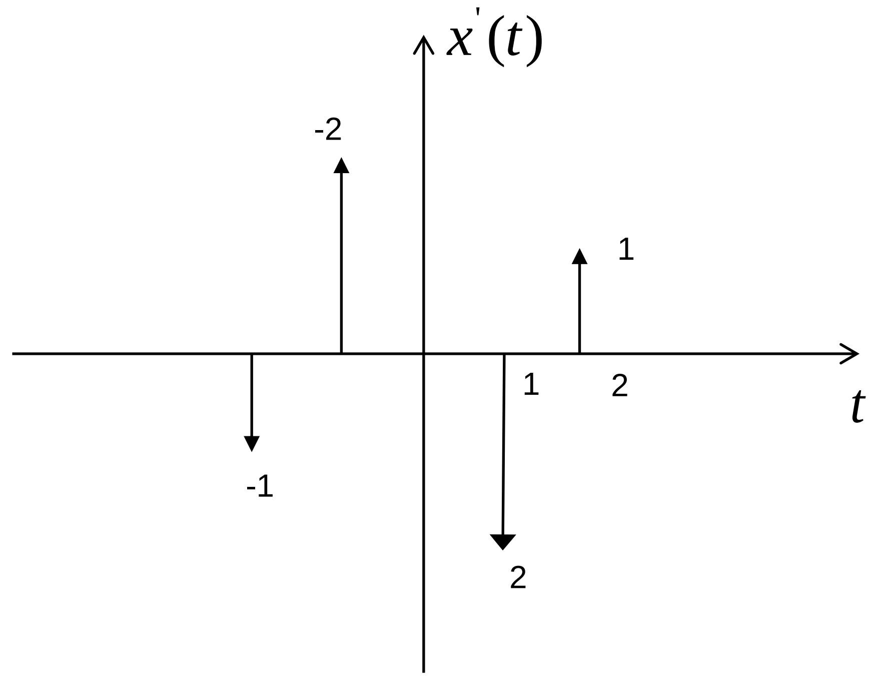


- **Solution 2:**

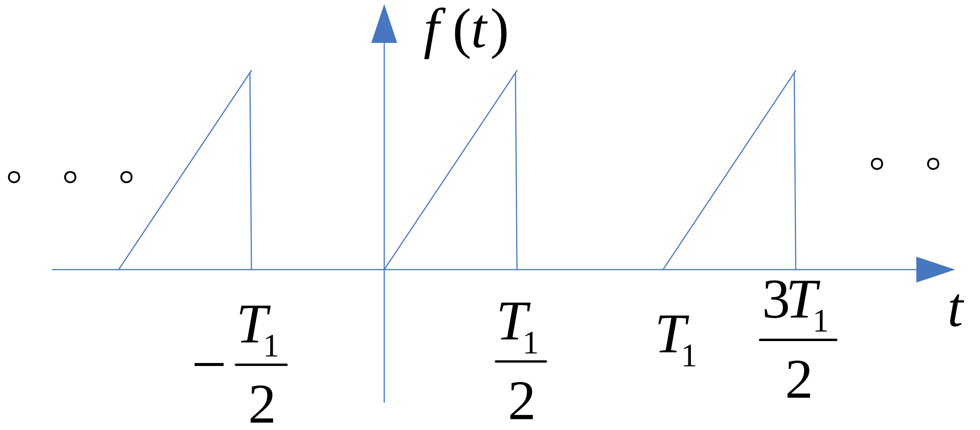
Assuming : $x_1(t) = x'(t)$

$$x_1(t) \leftrightarrow X_1(j\omega)$$

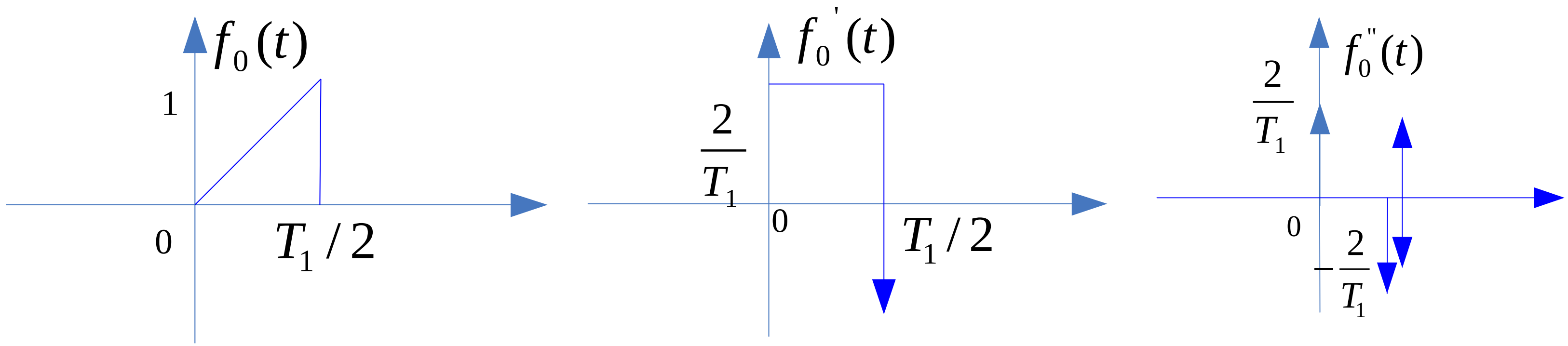
Then $$X(j\omega) = \frac{X_1(j\omega)}{j\omega} + \pi X_1(0)\delta(\omega) + 2\pi x(-\infty)\delta(\omega)$$



- Example: To determine the FC of the periodic signal by using FT



- Let $f_0(t) \leftrightarrow F_0(\omega)$ be the basic signal



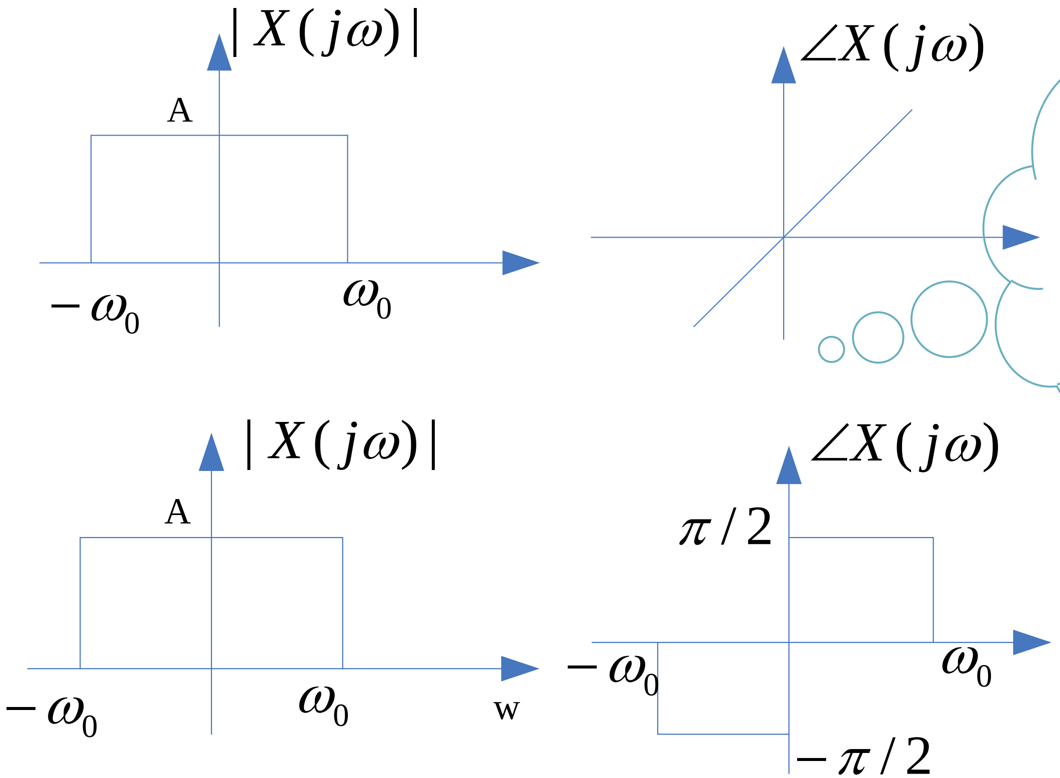
$$f_0''(t) = \frac{2}{T_1}\delta(t) - \frac{2}{T_1}\delta(t - \frac{T_1}{2}) - \delta'(t - \frac{T_1}{2})$$

- $F_n = \frac{1}{T_1} F_0(\omega) |_{\omega = n\omega_1}$

- Example :To determine x(t) according to $X(j\omega)$

1\.

2\.



Notes: They have different phase spectrum

Topic

- 4.0 Introduction
- 4.1 The Continuous-Time Fourier Transform
- 4.2 The Fourier Transform for Periodic Signals
- 4.3 Properties of the Continuous-Time Fourier Transform
- 4.4 The Convolution Property
- 4.5 The multiplication Property
- 4.6 System Characterized by Linear Constant-Coefficient Differential Equations

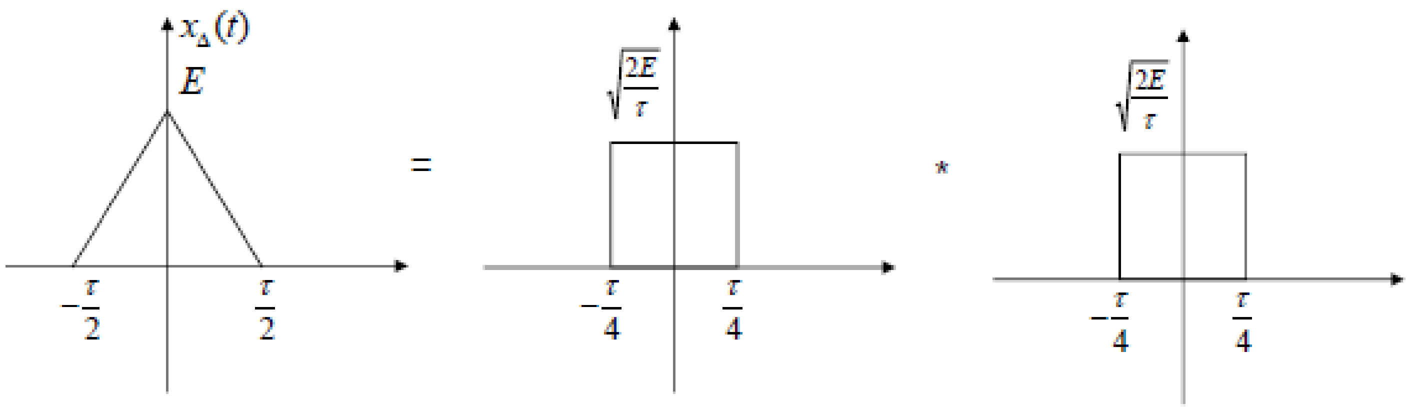
4.4.1 Convolution Property

$$x_1(t) \leftrightarrow X_1(j\omega) \quad x_2(t) \leftrightarrow X_2(j\omega)$$

$$\boxed{\begin{aligned} x(t) &= x_1(t) * x_2(t) \\ X(j\omega) &= X_1(j\omega) \cdot X_2(j\omega) \end{aligned}}$$

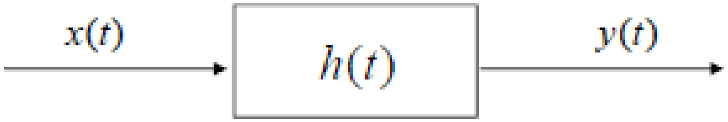
- Example: the Triangle Impulse Signal

$$x_{\Delta}(t)=\begin{cases} E(1-\dfrac{2|t|}{\tau}), & |t|\le\dfrac{\tau}{2} \\ 0, & |t|>\dfrac{\tau}{2} \end{cases}$$



4.4.2 Frequency Response

- Definition:



$$y(t) = x(t) * h(t)$$

$$x(t) \leftrightarrow X(j\omega) \quad y(t) \leftrightarrow Y(j\omega)$$

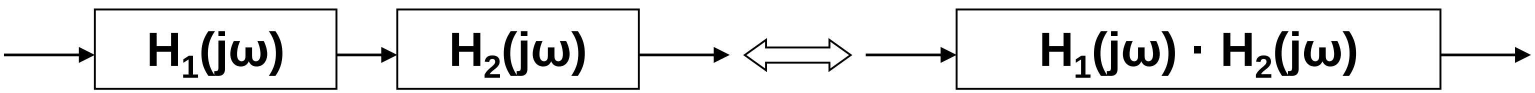
$$\boxed{H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt}$$ —frequency response

Conditioned on: $\int_{-\infty}^{\infty} |h(t)| dt < \infty$ —stable system

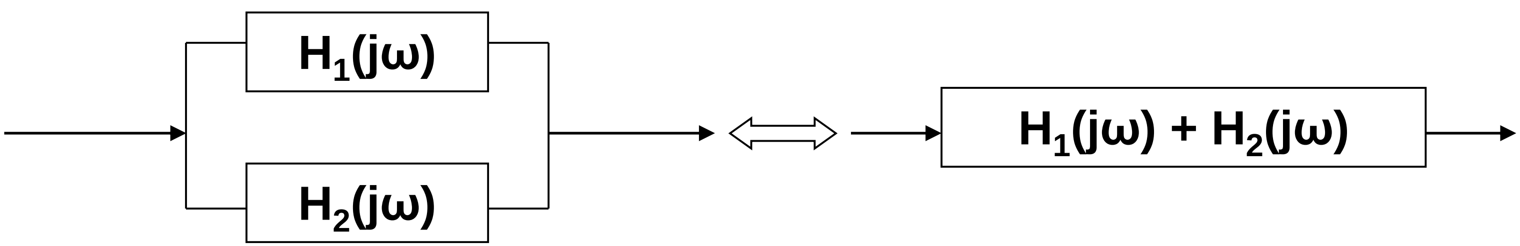
Then: $$\boxed{Y(j\omega) = X(j\omega) H(j\omega) \quad / \quad H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}}$$

4.4.2 Frequency Response

- The frequency response H(jω) can completely represent a stable LTI system (NOT all LTI systems)



Series interconnection of LTI systems (Cascaded system)



Parallel interconnection of LTI systems

4.4.2 Frequency Response

- The frequency response is the F.T. of the impulse response, it captures the change in complex amplitude of the Fourier transform of the input at each frequency ω

$$H(j\omega) = |H(j\omega)| e^{j\angle H(j\omega)}$$

Magnitude gain ($|H(j\omega)|$) — Phase shifting ($e^{j\angle H(j\omega)}$)

  - For a complex exponential input x(t), as a consequence of the eigenfunction property, the output y(t) can be expressed as:

$$x(t) = e^{j\omega_0 t} \rightarrow y(t) = H(j\omega)|_{\omega=\omega_0} \, e^{j\omega_0 t}$$

  - For a sinusoid input x(t), as a consequence of the eigenfunction property, the output y(t) can be expressed as:

$$x(t) = \cos(\omega_0 t) \rightarrow y(t) = |H(j\omega_0)| \cos(\omega_0 t + \angle H(j\omega_0))$$

- Example: Consider an LTI system with $H(j\omega) = \dfrac{1}{1 + j\omega}$
  If the input x(t)=sin(t), determine the output y(t)

- Solution:

$$\because H(j\omega) = \frac{1}{1 + j\omega}$$

$$\therefore |H(j\omega)| = \frac{1}{\sqrt{1+\omega^2}}$$

$$\angle H(j\omega) = \tan^{-1}(-\omega)$$

$$y(t) = |H(j1)| \sin(t + \angle H(j1))$$

$$= \frac{1}{\sqrt{2}} \sin\left(t - \frac{\pi}{4}\right)$$

- Example: Consider an LTI system with

$$h(t) = e^{-t}u(t)$$

for the input x(t)

$$x(t) = e^{-2t}u(t)$$

Determine the output of the system

$$y(t) = x(t) * h(t)$$

- Example: for a system with Gaussian response, i.e. the unit impulse response is Gaussian, consider the output of the system with a Gaussian input

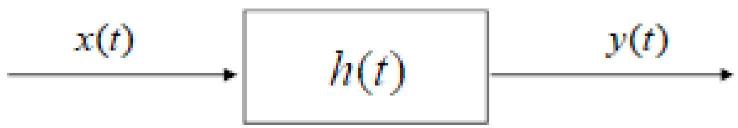
$$e^{-at^2} * e^{-bt^2} = \quad ? \qquad \sqrt{\frac{\pi}{a+b}} \cdot e^{-\left(\frac{ab}{a+b}\right)t^2}$$

$$\Downarrow$$

$$\sqrt{\frac{\pi}{a}}e^{-\frac{\omega^2}{4a}} \times \sqrt{\frac{\pi}{b}}e^{-\frac{-\omega^2}{4b}} = \frac{\pi}{\sqrt{ab}}e^{-\frac{\omega^2}{4}\left(\frac{1}{a}+\frac{1}{b}\right)}$$

Gaussian $\times$ Gaussian = Gaussian $\Rightarrow$ Gaussian $*$ Gaussian = Gaussian

Why: Log-Magnitude and Phase to illustrate the frequency response



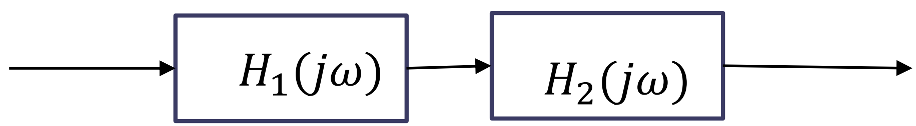
$$|Y(j\omega)| = |H(j\omega)| \times |X(j\omega)|$$

$$log|Y(j\omega)| = log|H(j\omega)| + log|X(j\omega)|$$

Easy to add

$$\angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$$

Cascading:



$$log|H(j\omega)| = log|H_1(j\omega)| + log|H_2(j\omega)|$$

$$\angle H(j\omega) = \angle H_1(j\omega) + \angle H_2(j\omega)$$

Easy to add

How: Plotting Log-Magnitude and Phase

- a) For real-valued signals and systems

$$\left.\begin{array}{rcl} |H(-j\omega)| & = & |H(j\omega)| \\ \angle H(-j\omega) & = & -\angle H(j\omega) \end{array}\right\} \Rightarrow$$

Plot for ω ≥ 0, often with a *logarithmic* scale for frequency in CT

- b) For historical reasons, log-magnitude is usually plotted in units of decibels (dB):

$$(1\ bel = 10\ decibels = \frac{\text{output power}}{\text{input power}} = 10)$$

Why 20 $\log_{10}(.)$

power, magnitude

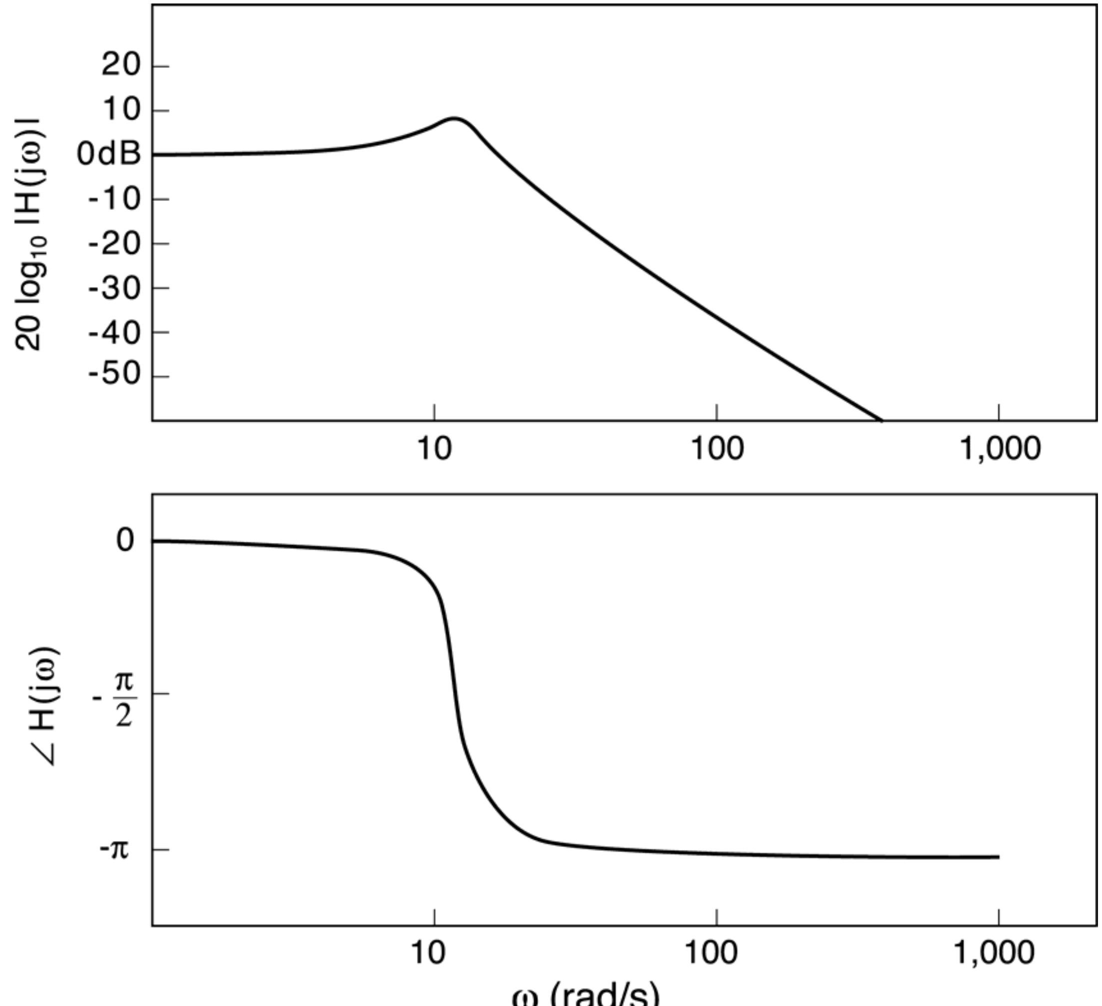
$$10\log_{10}|H(j\omega)|^2 = 20\log_{10}|H(j\omega)|$$

$$\begin{array}{rcr} |H(j\omega)| = 1 & \longrightarrow & 0\,\text{dB} \\ |H(j\omega)| = \sqrt{2} & \longrightarrow & \sim 3\,\text{dB} \\ |H(j\omega)| = 2 & \longrightarrow & \sim 6\,\text{dB} \\ |H(j\omega)| = 10 & \longrightarrow & 20\,\text{dB} \\ |H(j\omega)| = 100 & \longrightarrow & 40\,\text{dB} \end{array}$$

So… 20 dB or 2 *bels*:
= 10 amplitude gain
= 100 power gain

- **A Typical Bode plot for a second-order CT system**

$20\log_{10}|H(j\omega)|$ and $\angle H(j\omega)$ vs. $\log_{10}\omega$



4.4.3 Filtering

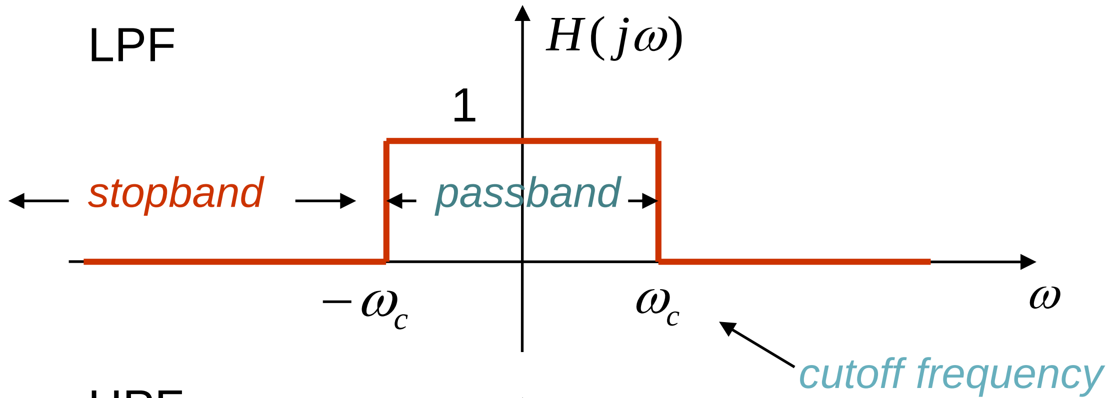
—a process in which the relative complex magnitudes of the frequency components in a signal are changed or some frequency components are completely eliminated

- Frequency-Selective Filters

—systems that are designed to pass some frequency components undistorted, and diminish/eliminate others significantly

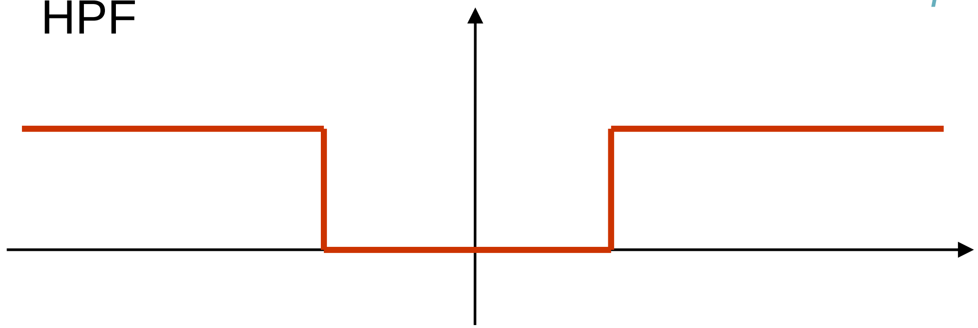
- Typical types of frequency-selective filters
  - LPF(Low-pass Filter)
  - HPF(High-pass Filter)
  - BPF(Band-pass Filter)
  - BSF (Band-stop Filter

LPF



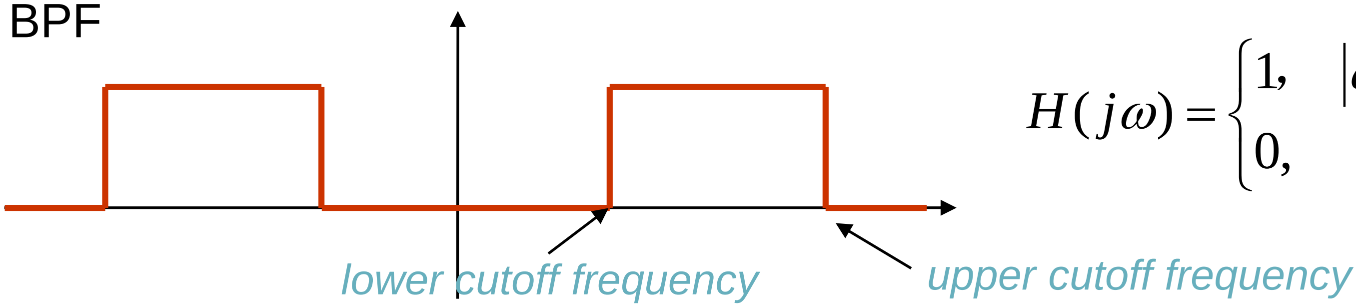
$$H(j\omega)=\begin{cases}1, & |\omega|\le\omega_c\\ 0, & |\omega|>\omega_c\end{cases}$$

HPF



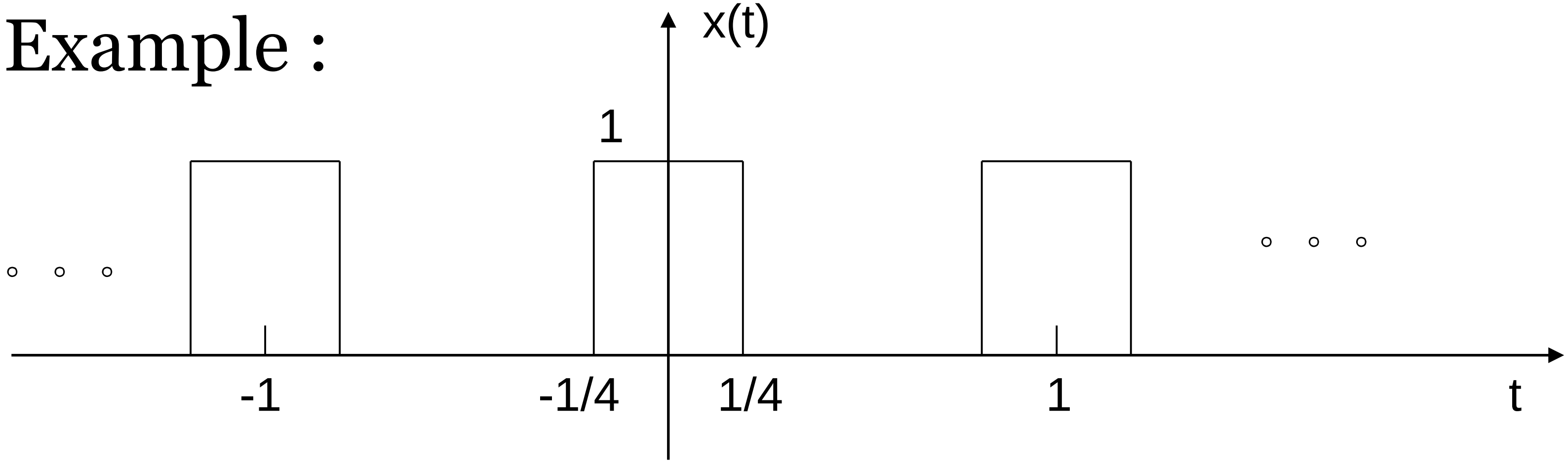
$$H(j\omega)=\begin{cases}1, & |\omega|>\omega_c\\ 0, & |\omega|\le\omega_c\end{cases}$$

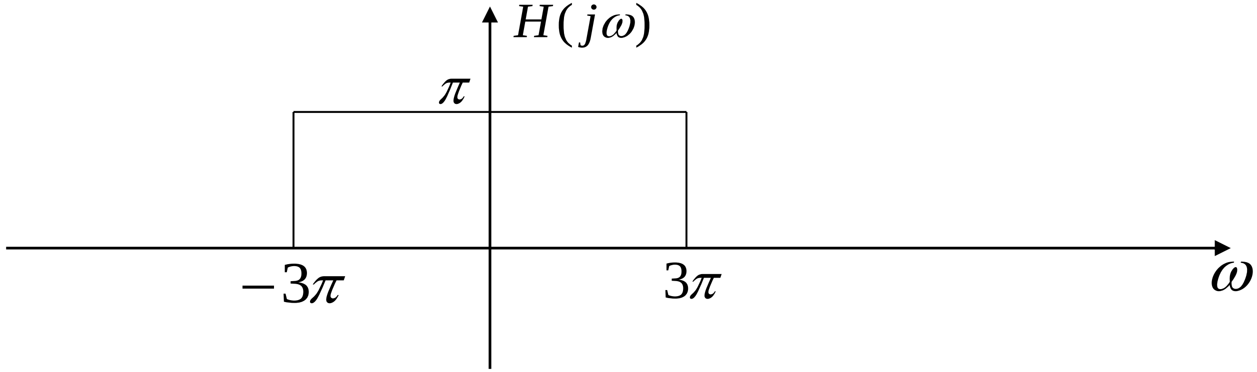
BPF



$$H(j\omega)=\begin{cases}1, & |\omega|\le\omega_{c1}, |\omega|\ge\omega_{c2}\\ 0, & \omega_{c1}<|\omega|<\omega_{c2}\end{cases}$$

- Example :





To determine the response of the LPF to the signal x(t)

- Some typical systems
  - ① Delay

$$\because y(t) = x(t - t_0)$$

$$\therefore Y(j\omega) = X(j\omega)e^{-j\omega t_0}$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = e^{-j\omega t_0}$$

  - ② Differentiator

$$\because y(t) = \frac{dx(t)}{dt}$$

$$\therefore Y(j\omega) = j\omega \cdot X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = j\omega$$

- ③ Integrator

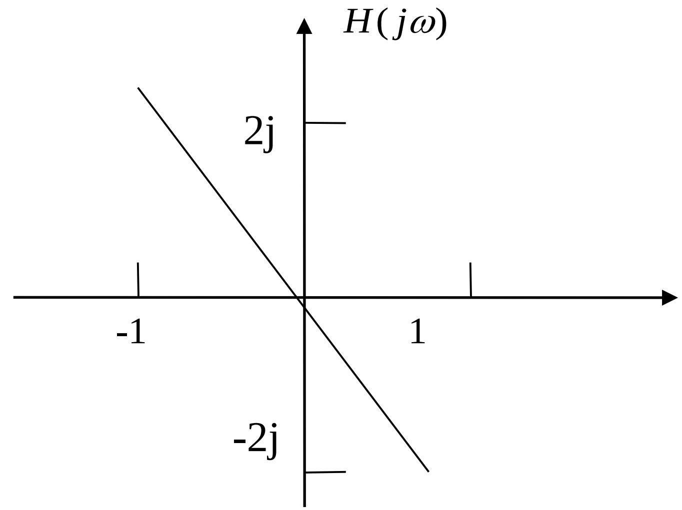
$$\because y(t)=\int_{-\infty}^{t}x(\tau)d\tau$$

$$\therefore Y(j\omega)=\frac{X(j\omega)}{j\omega}+\pi X(0)\delta(\omega)$$

when $X(0)=\int_{-\infty}^{\infty}x(t)dt=0$

$$H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{1}{j\omega}$$

- Example: to determine outputs of the system with H(jω) in the figure with the following input signals



1、 $x(t) = e^{jt}$

2、 $X(j\omega) = \dfrac{1}{(j\omega)(6 + j\omega)}$

- Example: for the following signal x(t) with period of 1

$$x(t)=\begin{cases}\sin 2\pi t, m\le t\le (m+\frac{1}{2})\\ 0,(m+\frac{1}{2})\le t\le m+1\end{cases}$$

$$H(j\omega)=\frac{j\omega}{3\pi}(-3\pi\le\omega<3\pi)$$

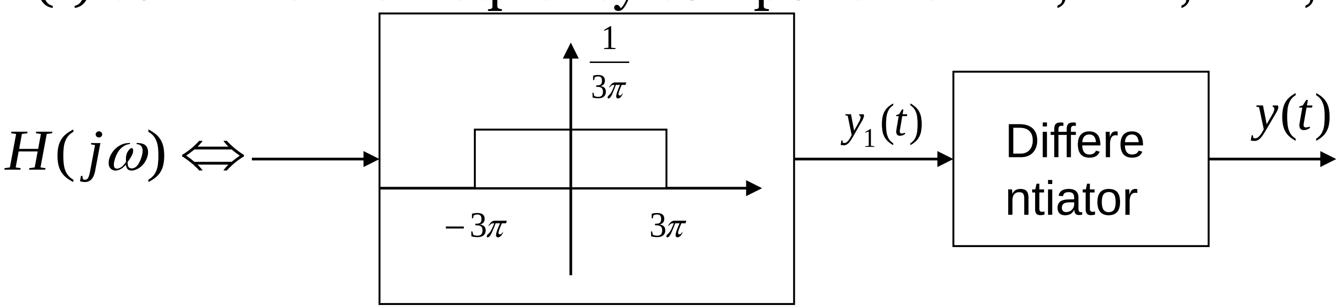
To determine the output of the system with frequency response H(jω) with the input x(t)

- Solution:

$$\because x(t) \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

and $\omega_0 = \dfrac{2\pi}{T} = 2\pi$

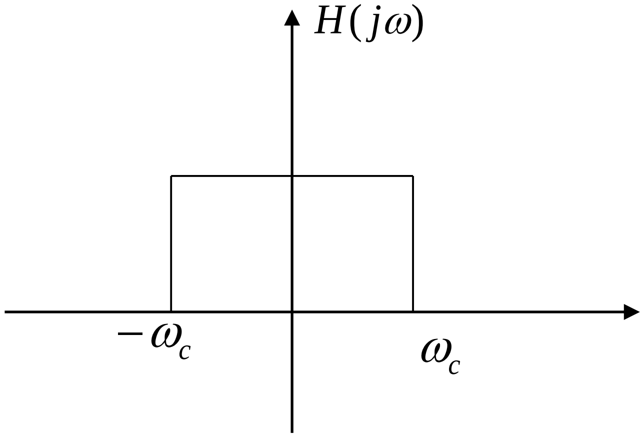
$\therefore$ x(t) contains the frequency components: $0, \pm 2\pi, \pm 4\pi, \cdots$



Only the DC and the first order harmonic components are within the passband of the LPF

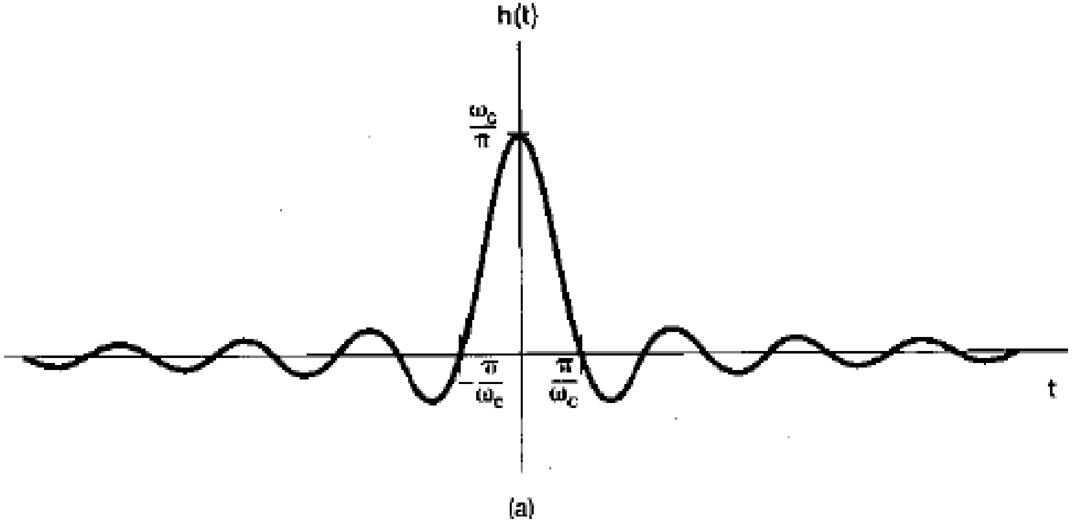
Filters

- Zero-phase shifting Ideal LPF

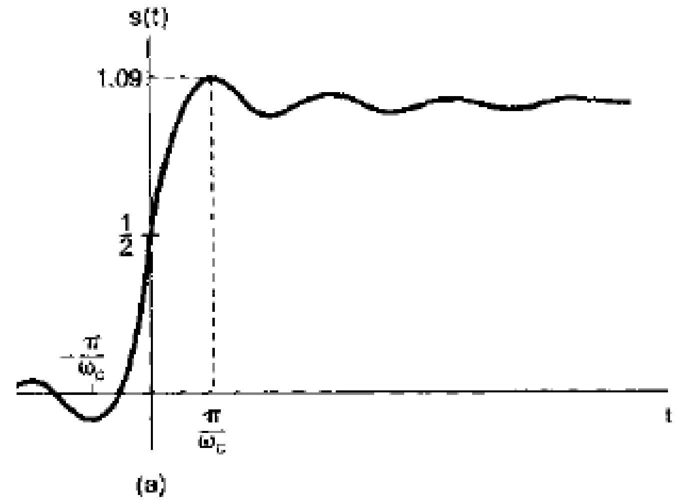


$$H(j\omega)=\begin{cases}1, & |\omega|<\omega_c \\ 0, & |\omega|>\omega_c\end{cases}$$

Unit impulse response



Unit step response



$$h(t) = \frac{\sin \omega_c t}{\pi t}$$

$$= \frac{\omega_c}{\pi} \cdot \frac{\sin \omega_c t}{\omega_c t}$$

$$s(t) = \frac{1}{2} + \frac{1}{\pi} Si(\omega_c t)$$

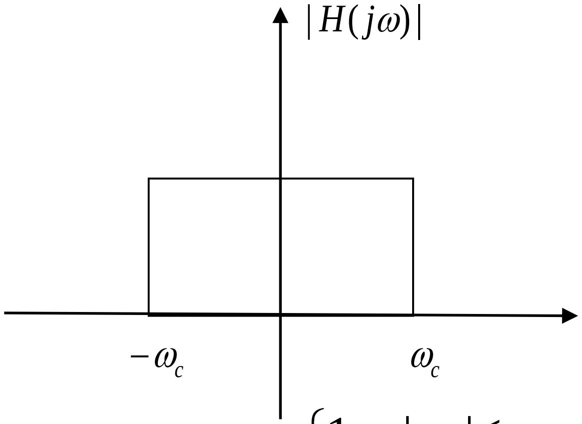
where

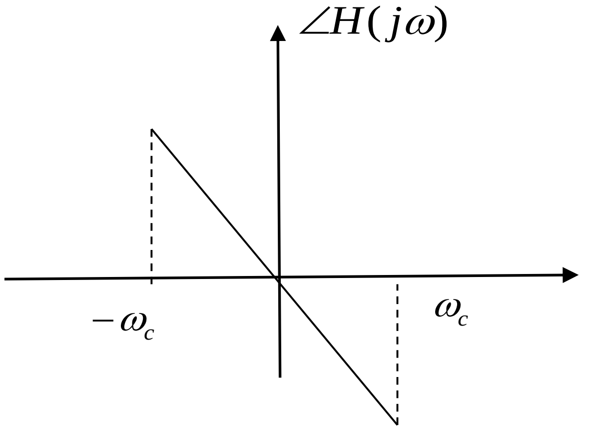
$$Si(y) = \int_{-\infty}^{\infty} \frac{\sin x}{x} dx$$

The unit impulse response of the HPF is $h(t) = \delta(t) - \dfrac{\sin \omega_c t}{\pi t}$

- # Linear Phase Ideal LPF

$$H(j\omega)=e^{-j\omega t_0}, \quad |\omega|\le\omega_c$$





$$|H(j\omega)|=\begin{cases}1, & |\omega|\le\omega_c\\ 0, & |\omega|>\omega_c\end{cases}$$

$$\angle H(j\omega)=-\omega t_0, \quad |\omega|\le\omega_c$$

$$Y(j\omega)=e^{-j\omega t_0}X(j\omega)\xleftrightarrow{\text{time-shift}}y(t)=x(t-t_0)$$

Result: Linear phase ⇔ simply a rigid shift in time, *no* distortion
Nonlinear phase ⇔ distortion as well as shift

- Unit impulse response:

$$h(t) = \frac{\sin \omega_c (t - t_0)}{\pi (t - t_0)}$$

- Unit step response:

$$s(t) = \frac{1}{2} + \frac{1}{\pi} Si[\omega_c (t - t_0)]$$

- How do we think about signal delay when the phase is nonlinear?

Concept of Group Delay

When the signal is narrow-band and concentrated near ω_0, $\angle H(jw)$ ~ linear with ω near ω_0, then the differential of $\angle H(jw)$ at ω_0 reflects the time delay.

For frequencies “near” ω_0

$$\angle H(j\omega) \approx \angle H(j\omega_0) - \tau(\omega_0)(\omega - \omega_0) = \phi - \tau(\omega_0)\cdot\omega$$

$$\tau(\omega) = -\frac{d}{d\omega}\{\angle H(j\omega)\} = \text{ Group Delay}$$

$$\Downarrow$$

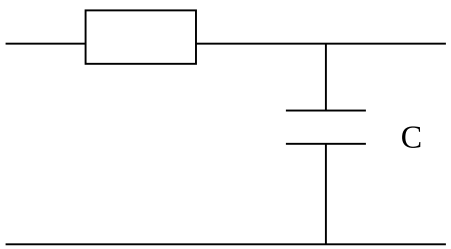
For w “near” ω_0

$$H(j\omega) \approx |H(j\omega_0)|e^{j\phi}e^{-j\tau(\omega_0)\omega}$$

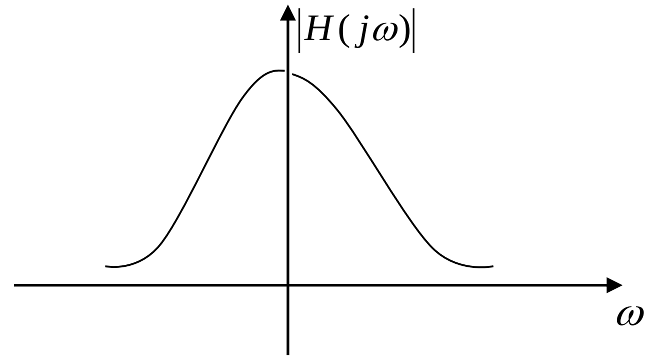
$$\Rightarrow \quad e^{j\omega t} \longrightarrow \sim |H(j\omega)|e^{j\phi}e^{j\omega(t-\tau(\omega_0))}$$

$T(\omega_0)$ Time delay of the original signal

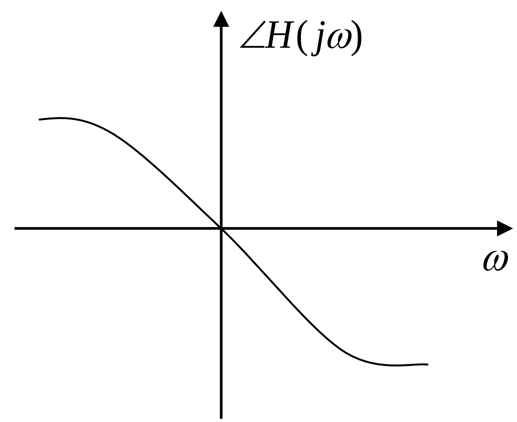
- Non-ideal LPF



$$H(j\omega) = \frac{\alpha}{\alpha + j\omega}, \quad \alpha = \frac{1}{RC}$$

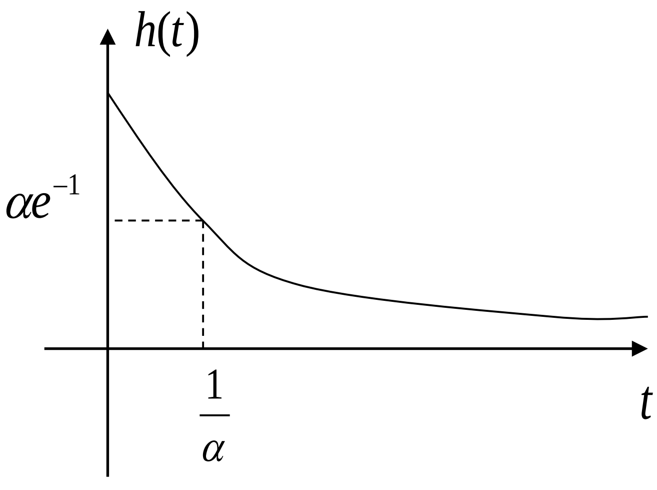


$$|H(j\omega)| = \frac{\alpha}{\sqrt{\alpha^2 + \omega^2}}$$

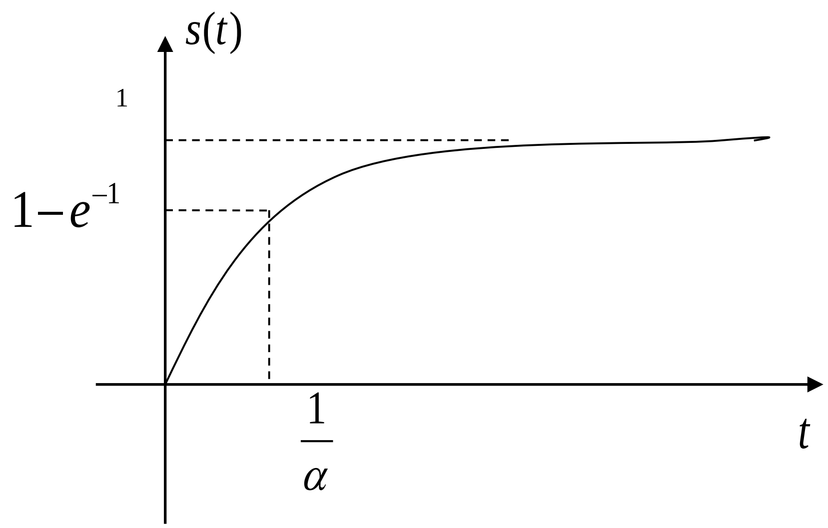


$$\angle H(j\omega) = \tan^{-1}\left(\frac{\omega}{\alpha}\right)$$

Unit impulse response



Unit step response



$$h(t) = \alpha e^{-\alpha t} u(t)$$

$$s(t) = (1 - e^{-\alpha t}) u(t)$$

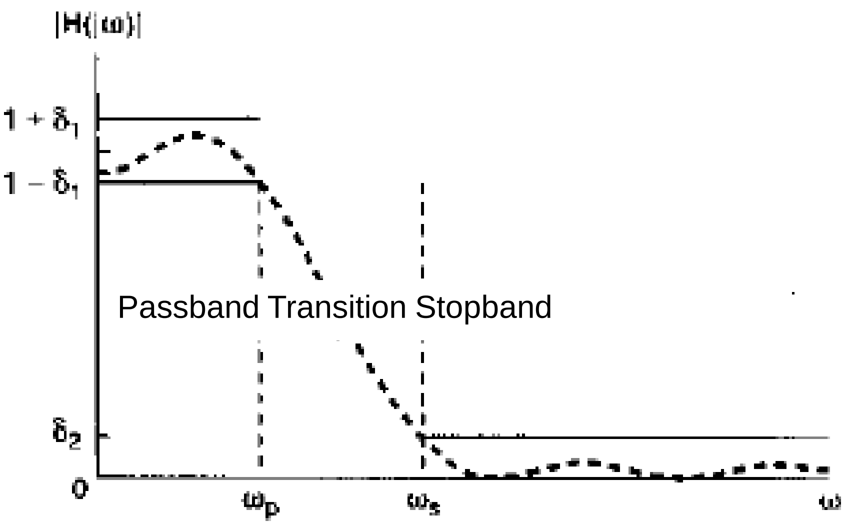
- causal $h(t<0) = 0$, decaying
- s(t) non-oscillation and non-overshoot

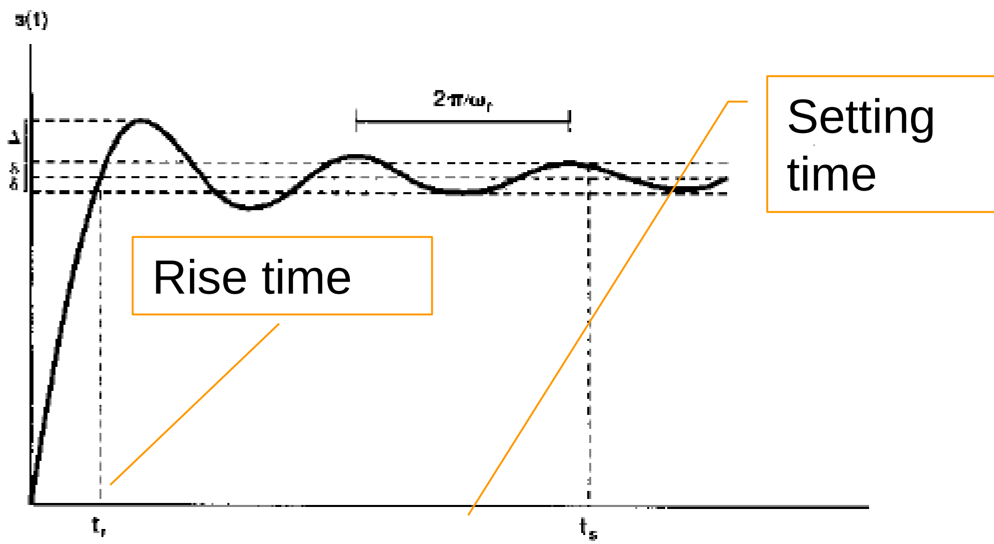
- Time domain and frequency domain aspects of non-ideal filter

Definitions:
Passband ripple: δ_1
Stopband ripple: δ_2

Definitions:
Rise time: t_r
Setting time: t_s
Overshoot: Δ
Ringing frequency ω_r

  - Trade-offs between time domain and frequency domain characteristics, i.e. the width of transition band ↔ the setting time of the step response





Setting time: the time at which the step response settles to within δ (a specified tolerance) of its final value

Topic

- 4.0 Introduction
- 4.1 The Continuous-Time Fourier Transform
- 4.2 The Fourier Transform for Periodic Signals
- 4.3 Properties of the Continuous-Time Fourier Transform
- 4.4 The Convolution Property
- 4.5 The multiplication Property
- 4.6 System Characterized by Linear Constant-Coefficient Differential Equations

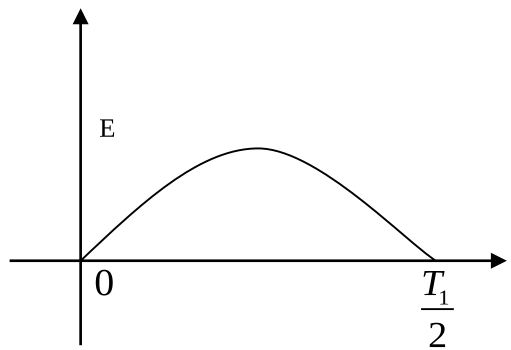
4.5.1 Multiplication Property

$$x_1(t) \leftrightarrow X_1(j\omega) \quad x_2(t) \leftrightarrow X_2(j\omega)$$

$$x(t) = x_1(t) \cdot x_2(t)$$

$$X(j\omega) = \frac{1}{2\pi}[X_1(j\omega) * X_2(j\omega)]$$

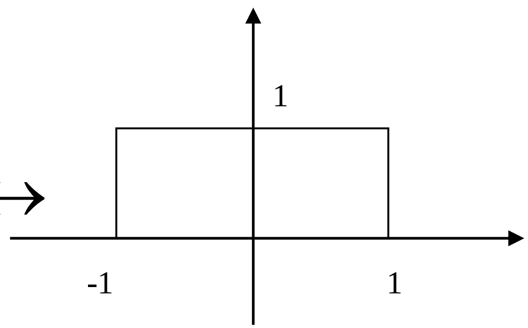
- Example :



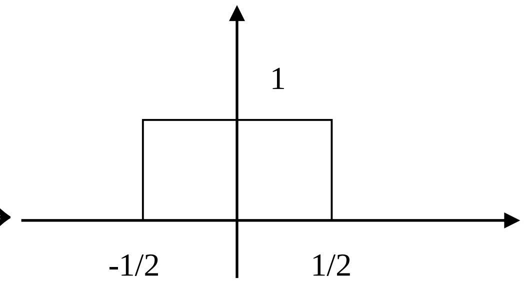
$$x(t) = E \sin \omega_1 t [u(t) - u(t - \frac{T_1}{2})]$$

- Example :

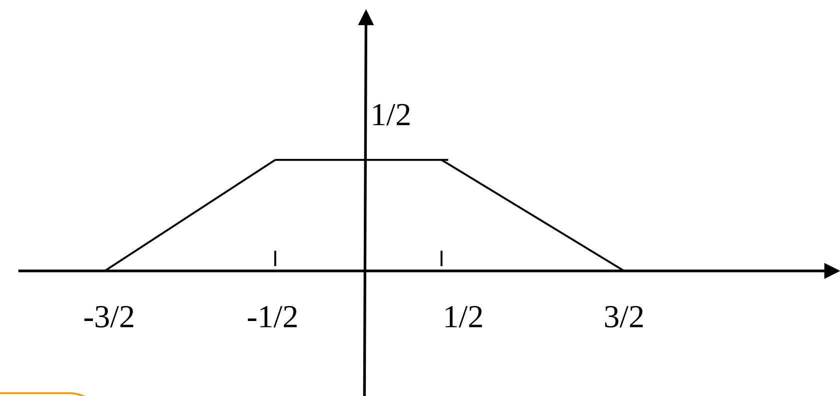
$$\because \frac{\sin t}{\pi t} \leftrightarrow$$



$$\frac{\sin \frac{t}{2}}{\pi t} \leftrightarrow$$



$$\therefore x(t) \leftrightarrow \frac{1}{2\pi} \cdot \pi \{ \Im[\frac{\sin t}{\pi t}] * \Im[\frac{\sin \frac{t}{2}}{\pi t}]\} =$$



Then

$$\int \frac{\sin t \cdot \sin \frac{t}{2}}{\pi t^2} dt = ?$$

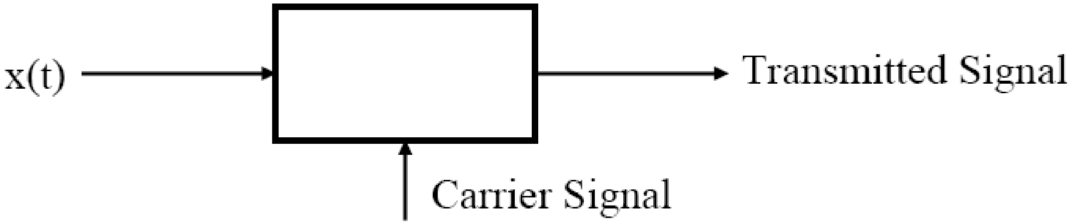
- Example :

$$r(t) = s(t) \times P(t) \leftrightarrow R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)]$$

$$For\ \ p(t) = cos\omega_0 t \leftrightarrow P(j\omega) = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

$$R(j\omega) = \frac{1}{2}[S(j(\omega - \omega_0)) + S(j(\omega + \omega_0))]$$

4.5.2 Modulation

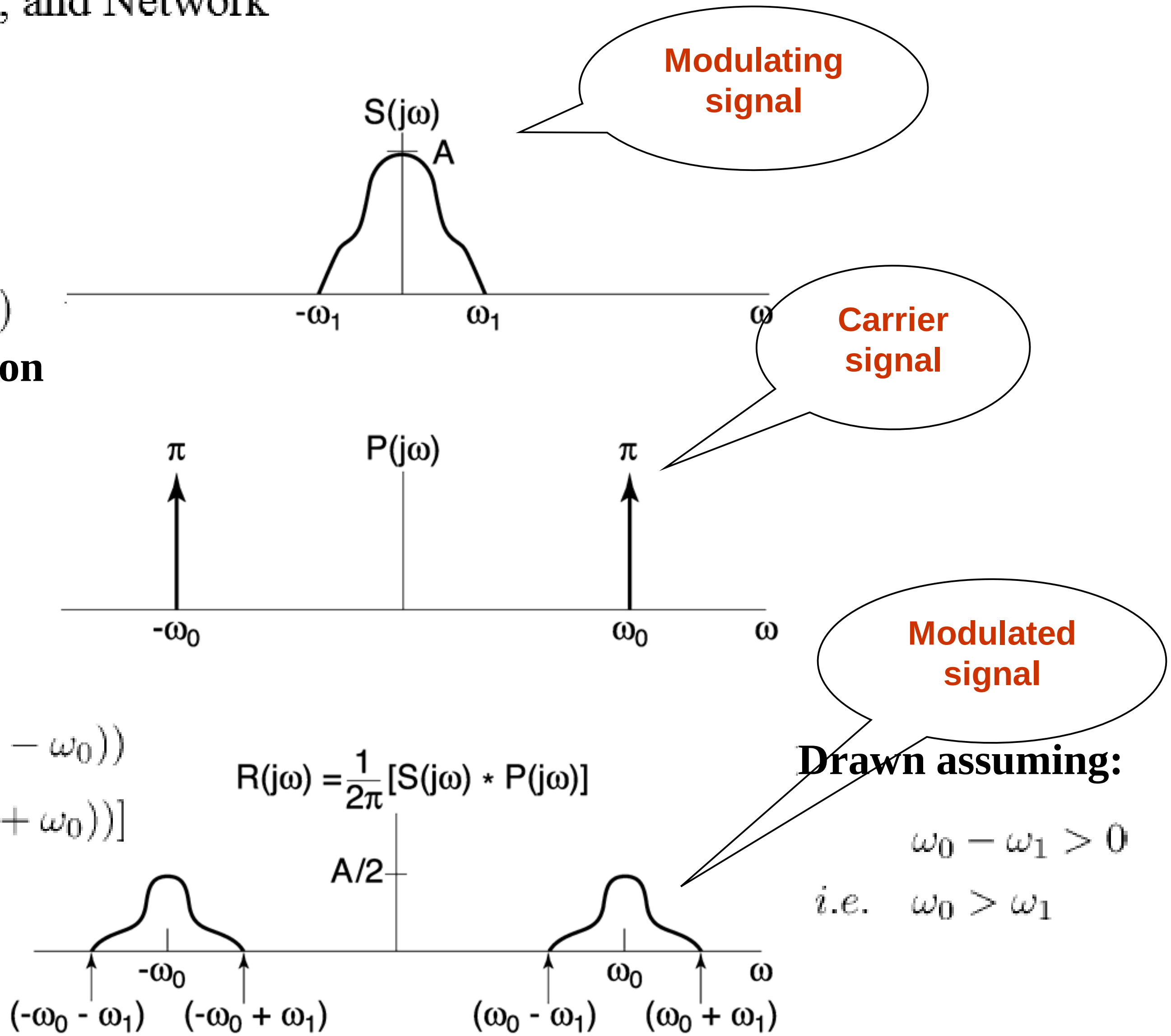


- Why?
  - More efficient to transmit E&M signals at higher frequencies
  - Transmitting multiple signals through the same medium using different carriers
  - Transmitting through "channels" with limited passbands
  - Others…
- How?
  - Many methods
  - Focus here for the most part on Amplitude Modulation (AM)

$$r(t) = s(t) \cdot \cos(\omega_0 t)$$

Amplitude modulation (AM)

$$R(j\omega) = \frac{1}{2}[S(j(\omega - \omega_0)) + S(j(\omega + \omega_0))]$$



Drawn assuming:

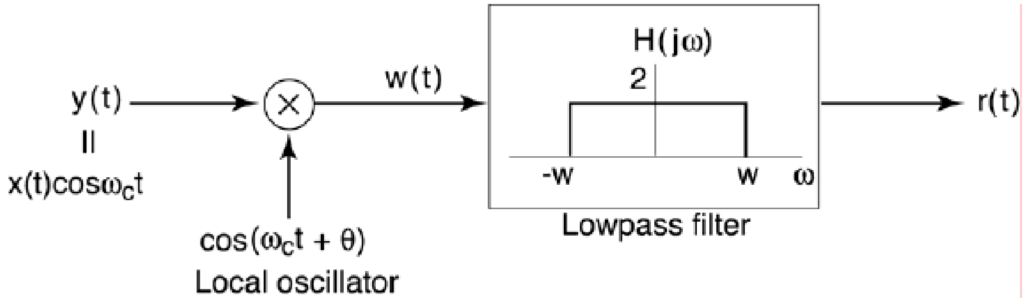
$$\omega_0 - \omega_1 > 0$$

$$i.e. \quad \omega_0 > \omega_1$$

• Synchronous Demodulation of Sinusoidal AM

If $\theta = 0$

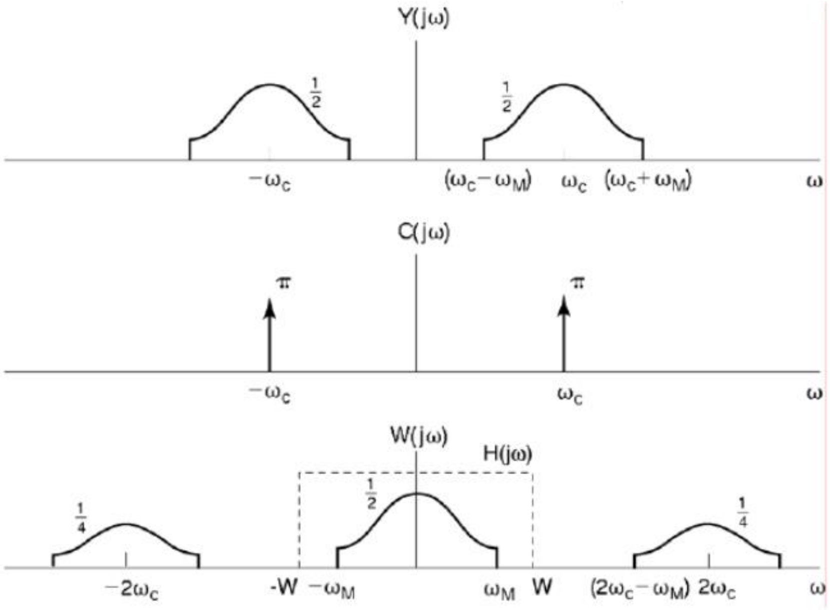
$$\hat{x}(t) = x_c(t) \times c(t)$$



$$\hat{X}(j\omega) = \frac{1}{2\pi}[X_c(j\omega) * C(j\omega)]$$

$$= \frac{1}{2}\left[X_c\big(j(\omega - \omega_0)\big) + X_c\big(j(\omega + \omega_0)\big)\right]$$

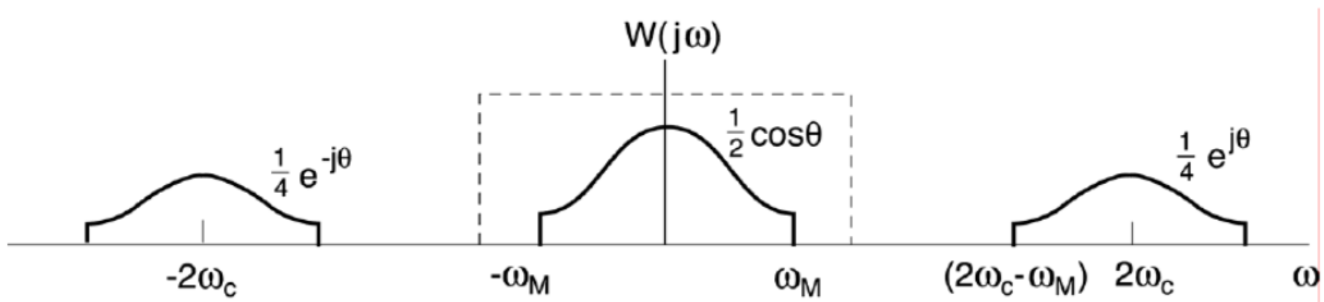
$$= \frac{1}{2}X(j\omega) + \frac{1}{4}[X_c(j(\omega - 2\omega_0)) + X_c(j(\omega + 2\omega_0))]$$



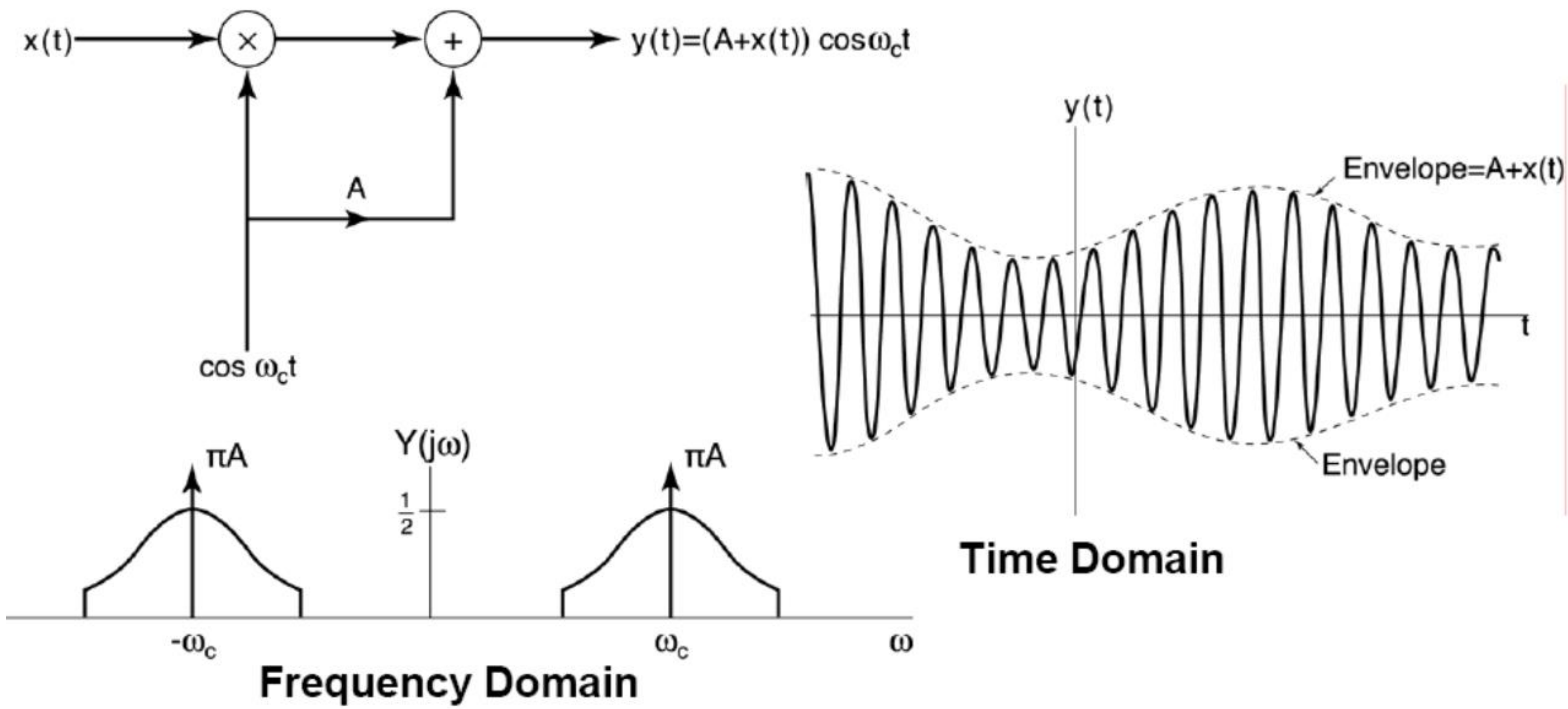
What if $\theta \neq 0$?

- Synchronous Demodulation (with phase error) in the Frequency Domain

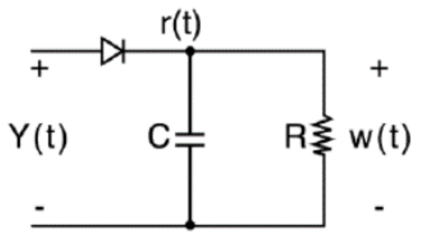
$$cos(\omega_c t+\theta) \leftrightarrow \pi e^{j\theta}\delta(\omega-\omega_0)+\pi e^{-j\theta}\delta(\omega+\omega_0)$$

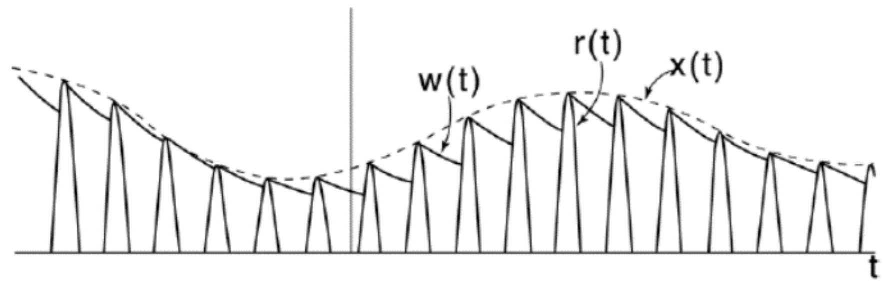


- Asynchronous Demodulation
  - Assume $\omega_c >> \omega_M$, so signal envelope looks like x(t)
  - Add same carrier with amplitude A to signal



$A = 0 \Rightarrow$ DSB/SC (Double Side Band, Suppressed Carrier)
$A > 0 \Rightarrow$ DSB/WC (Double Side Band, With Carrier)





In order for it to function properly, the envelope function mustbe positive for all time, *i.e.*$A+x(t) > 0$ for all t.

Demo: Envelope detection for asynchronous demodulation.

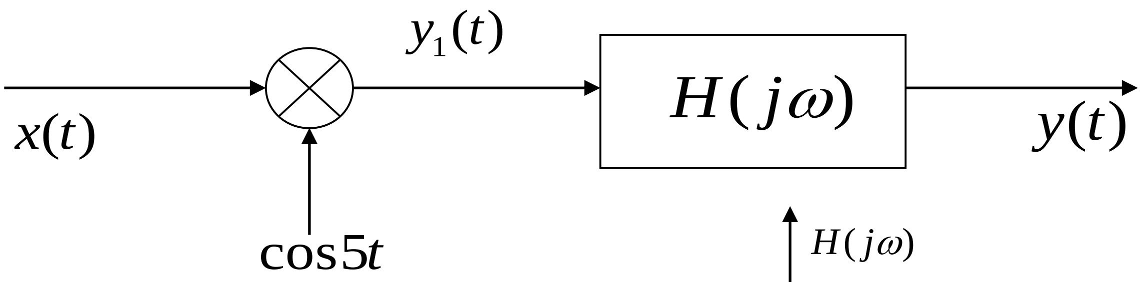
Advantages of asynchronous demodulation:

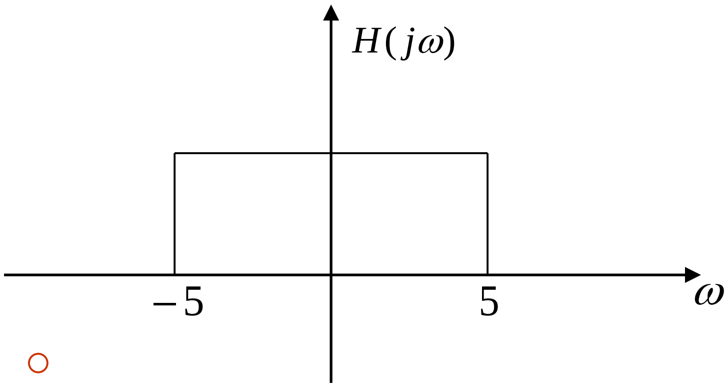
— Simpler in design and implementation.

Disadvantages of asynchronous demodulation:

— Requires extra transmitting power $[A\cos\omega_c t]^2$to make sure $A+x(t) > 0 \Rightarrow$Maximum power efficiency = 1/3 (P8.27)

- Example:





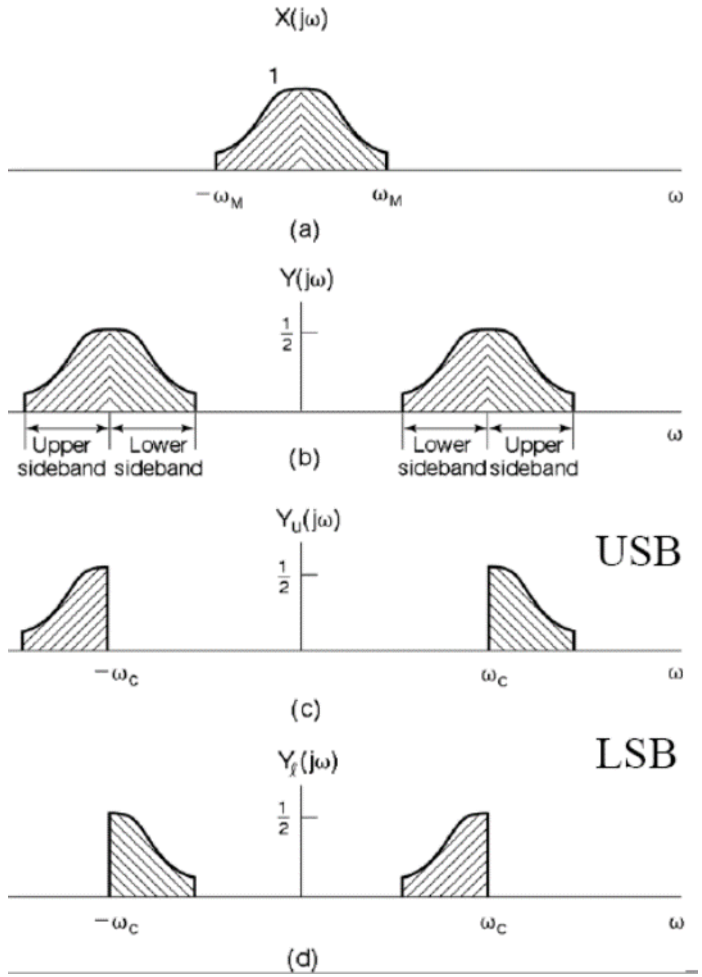
For $x(t) = \dfrac{\sin 2t}{\pi t}$

To determine $y(t)$

Signal processing in frequency domain

- Double-Sideband (DSB) and Single-Sideband (SSB) AM

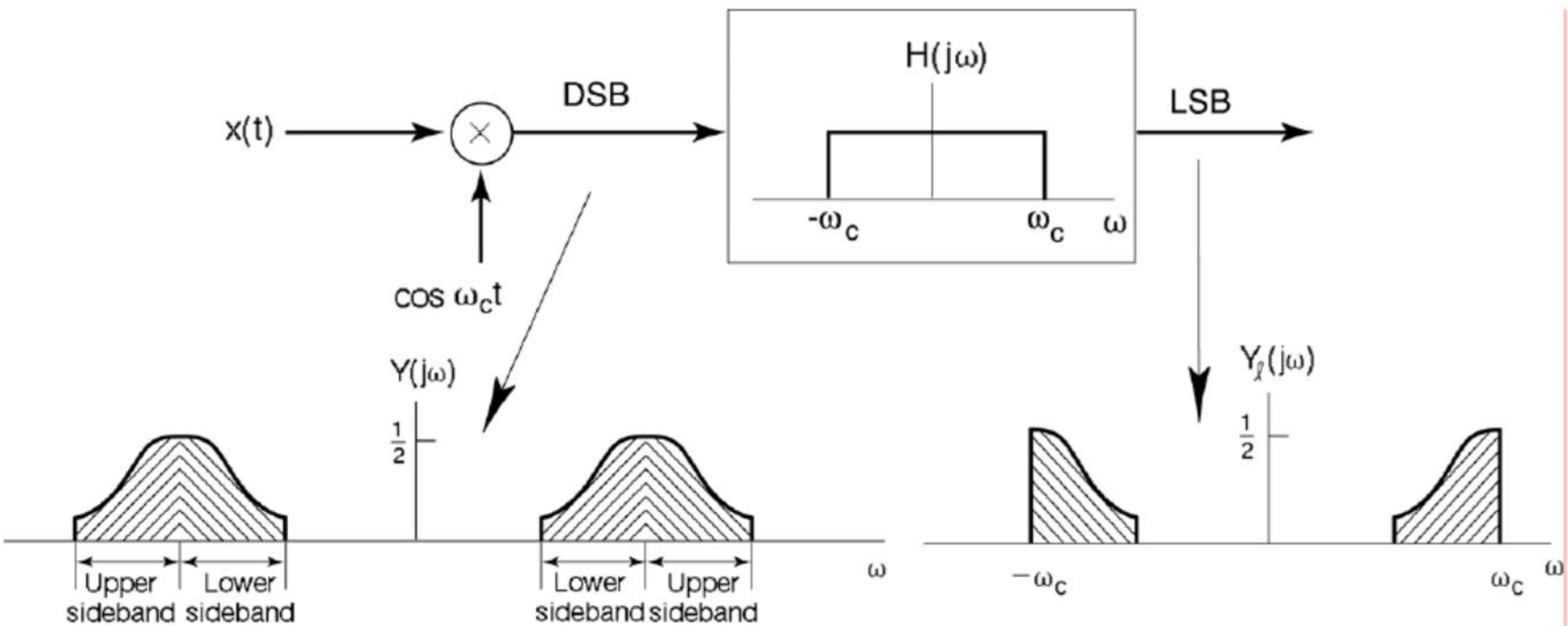
Since *x*(*t*) and *y*(*t*) are *real*, from Conjugate symmetry both *LSB* and *USB* signals carry exactly the same information.



DSB, occupies **2**ω_M bandwidth in $\omega > 0$.

Each sideband approach only occupies ω_M bandwidth in $\omega > 0$.

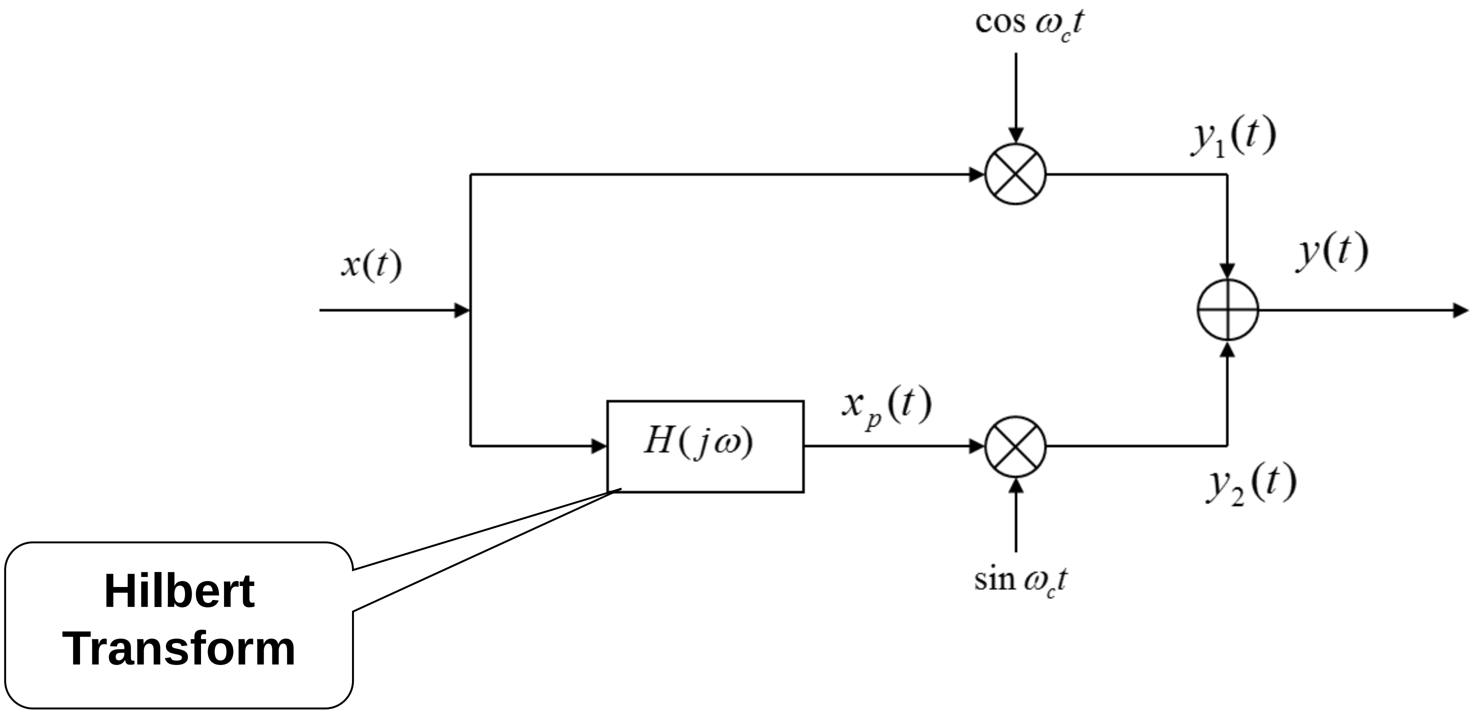
- Single-Sideband (SSB) AM



Can also get SSB/SC

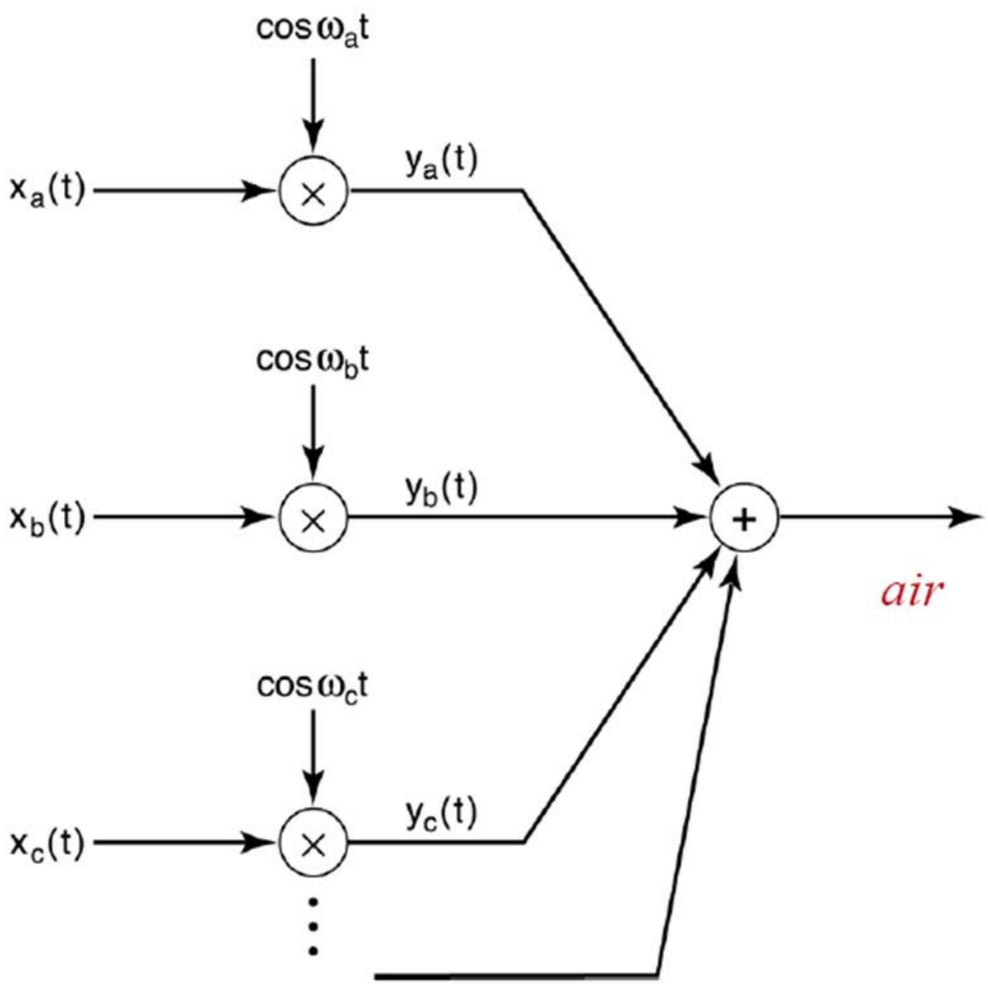
or SSB/WC

- An implementation of SSB modulation, p600,figure 8.21-22



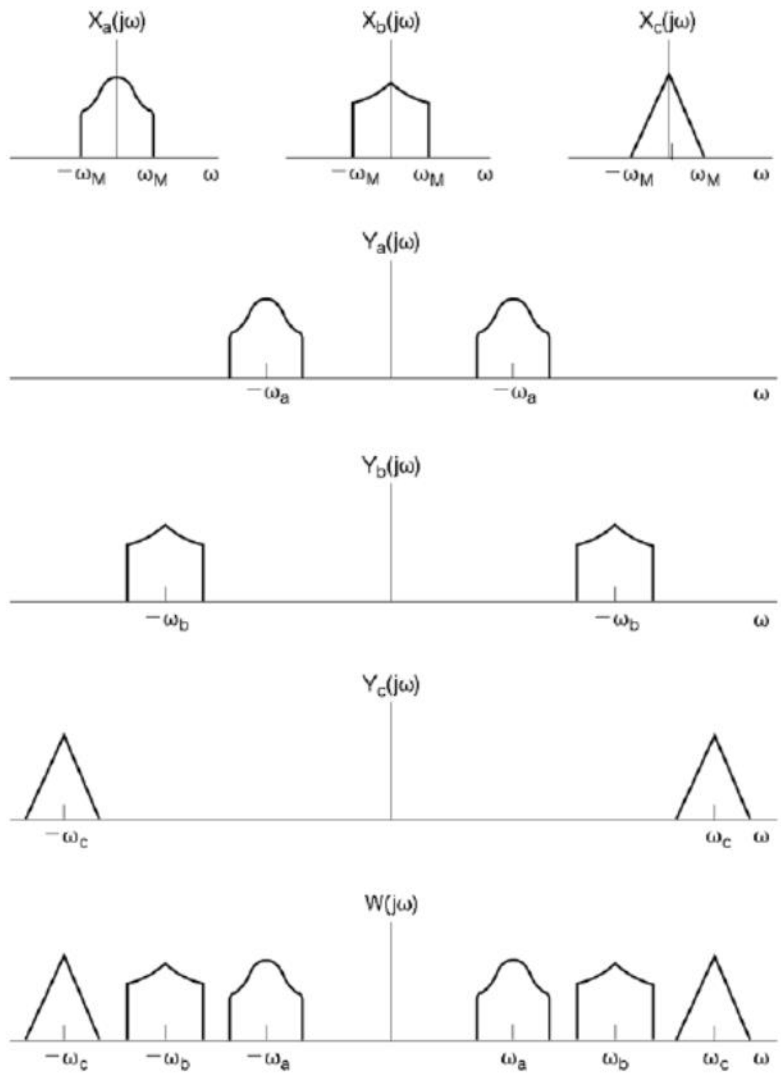
$$H(j\omega) = \begin{cases} -j & \omega > 0 \\ +j & \omega < 0 \end{cases} \leftrightarrow \quad h(t) = \frac{1}{\pi t}$$

- Frequency-Division Multiplexing (FDM)
  (Examples: Radio-station signals and analog cell phones)



All the channels can share the same medium.

- FDM in the Frequency-Domain



"Baseband" signals

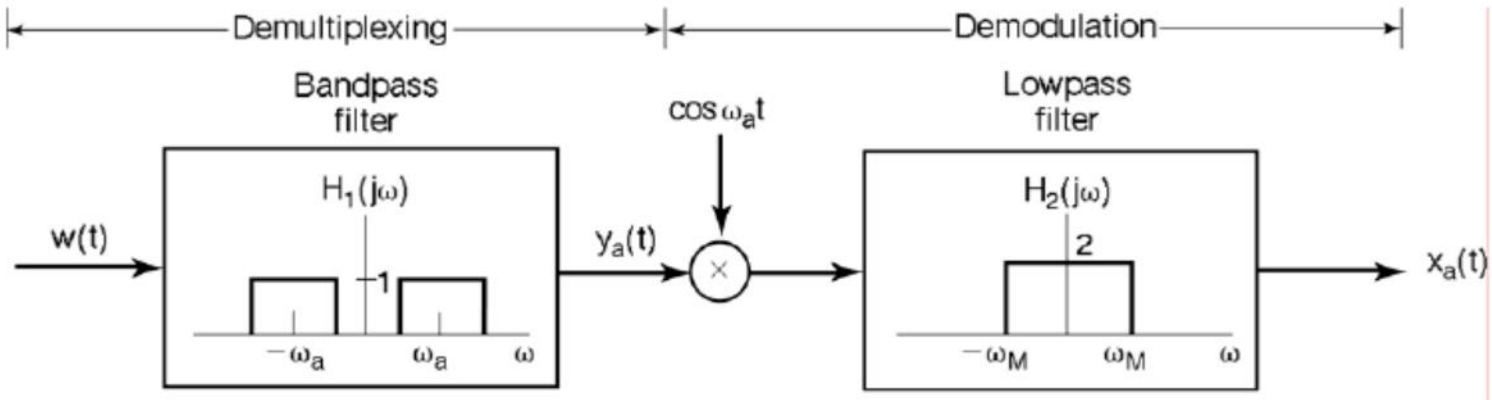
Channel a

Channel b

Channel c

Multiplexed signals

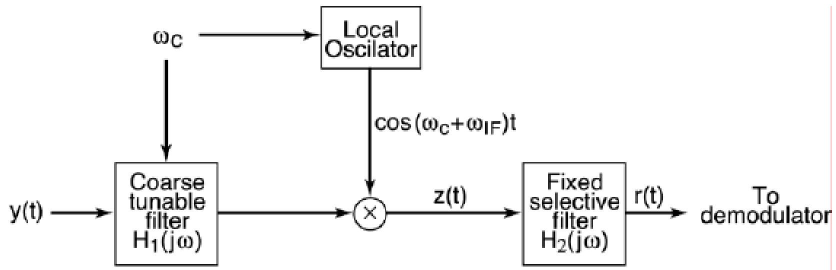
- Demultiplexing and Demodulation

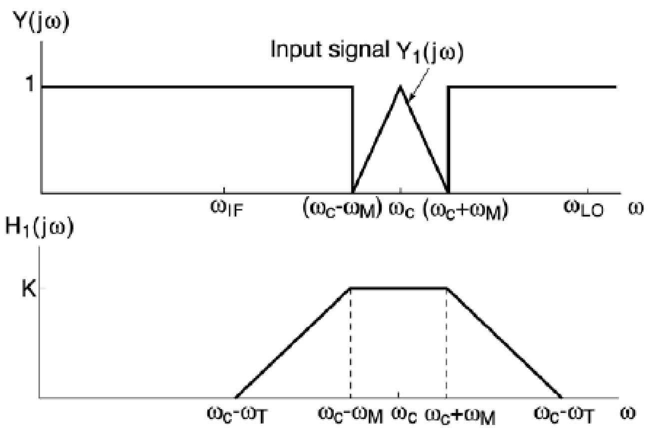


ω_a needs to be tunable

- Channels must not overlap ⇒Bandwidth Allocation
- It is difficult (and expensive) to design a highly selective bandpass filter with a tunable center frequency
- Solution –Superheterodyne Receivers

• The Superheterodyne Receiver

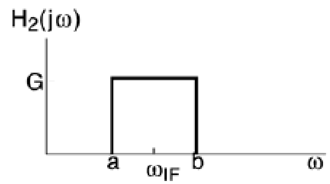




Coarse tunable filter

$$\text{AM,}\quad \frac{\omega_c}{2\pi} = 535 - 1605\, kHz \;\text{— RF}$$

$$\text{FCC:}\quad \frac{\omega_{IF}}{2\pi} = 455\, kHz \;\text{— IF}$$



Fixed BPF

- Operation principle:

— Down convert from ωc to ω_{IF}, and use a coarse tunable BPF for the front end.

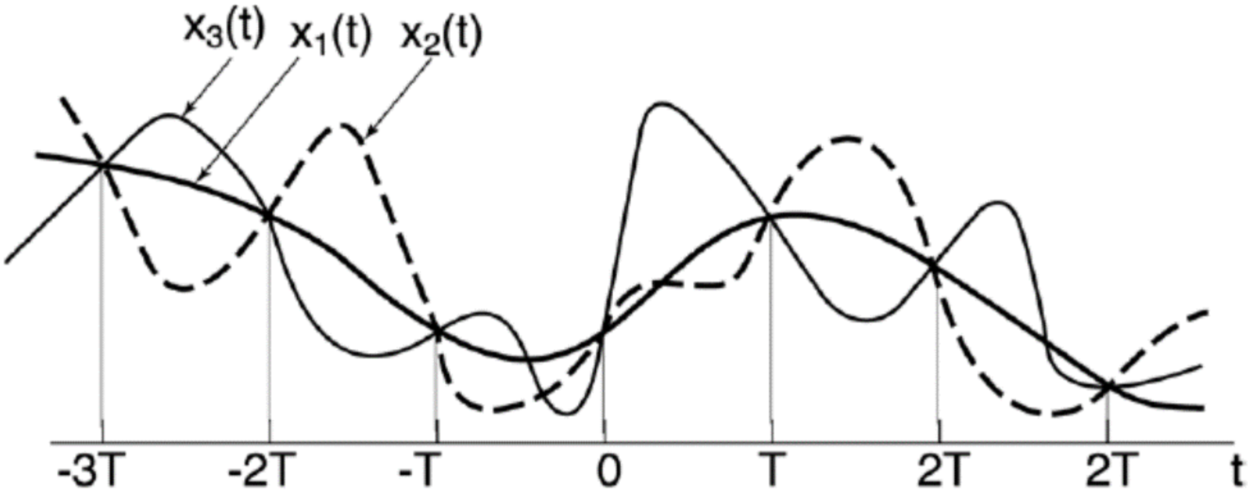
— Use a sharp-cutoff fixed BPF at ω_{IF} to get rid of other signals.

4.5.3 Sampling

- Most of the signals we encounter are CT signals, e.g. x(t). How do we convert them into DT signals x[n] to take advantages of the rapid progress and tools of digital signal processing
  - — Sampling, taking snap shots of x(t) every T seconds
- T –sampling period, $x[n] \equiv x(nT)$, n= ..., -1, 0, 1, 2, ... — Regularly spaced samples
- Applications and Examples
  - —Digital Processing of Signals
  - —Images in Newspapers
  - —Sampling Oscilloscope
  - —...

How do we perform sampling?

- Why/When Would a Set of Samples Be Adequate?
  - Observation: Lots of signals have the same samples



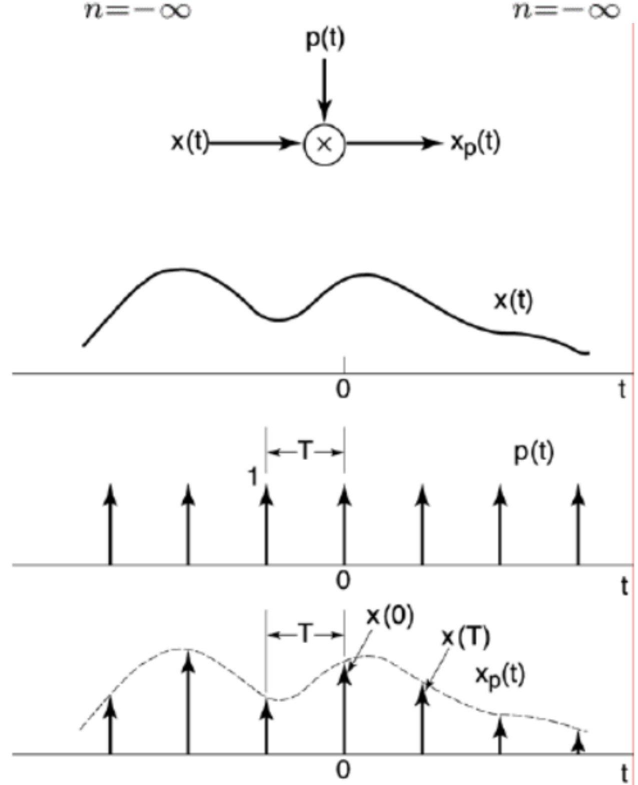
  - By sampling we throw out lots of information –all values of x(t) between sampling points are lost.
  - Key Question for Sampling:

  Under what conditions can we reconstruct the original CT signal x(t) from its samples?

- Impulse Sampling—Multiplying x(t) by the sampling function

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_p(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$



- Analysis of Sampling in the Frequency Domain

$$x_p(t) = x(t) \cdot p(t)$$

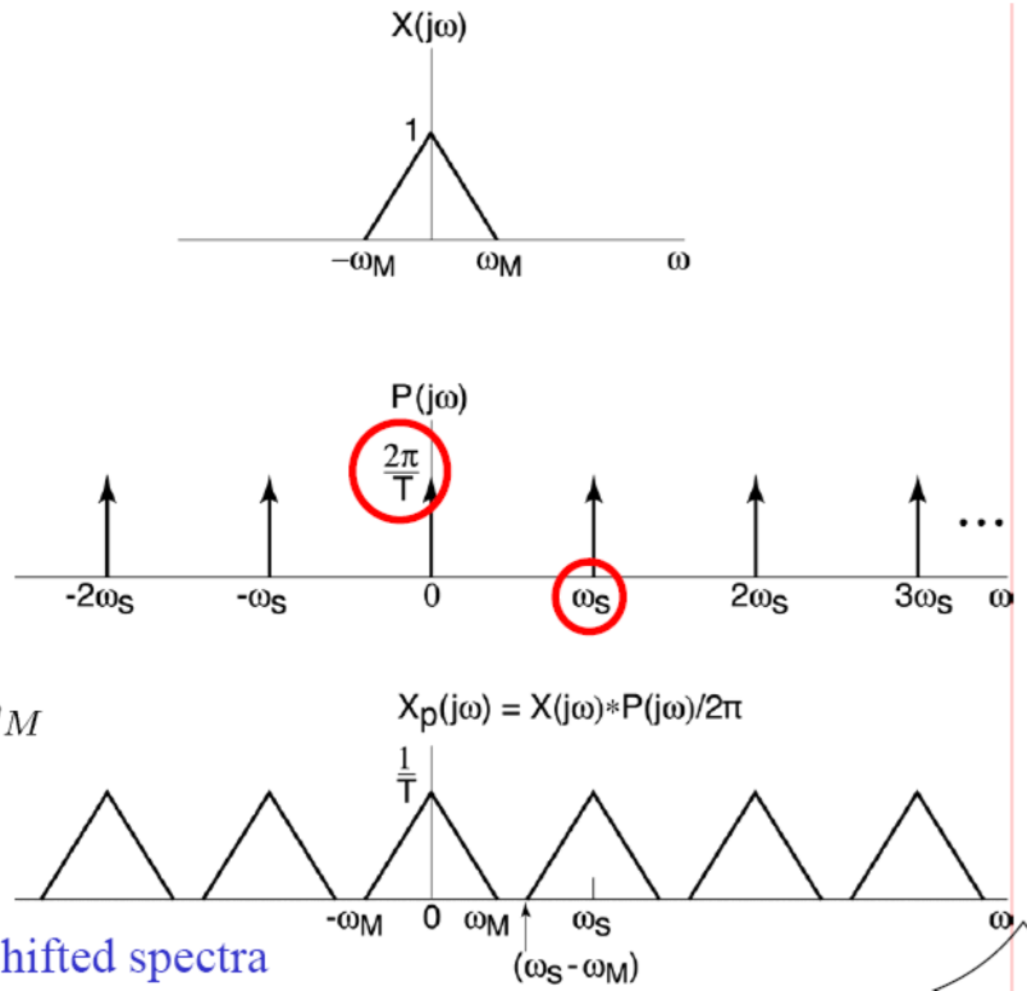
Multiplication Property => $X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$

$$P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

$\omega_s = \frac{2\pi}{T}$ =Sampling Frequency

$$\begin{aligned} X_p(j\omega) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j\omega) * \delta(\omega - k\omega_s) \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)) \end{aligned}$$

- Illustration of sampling in the frequency-domain for a band-limited ($X(j\omega)=0$ for $|\omega| > \omega M$) signal



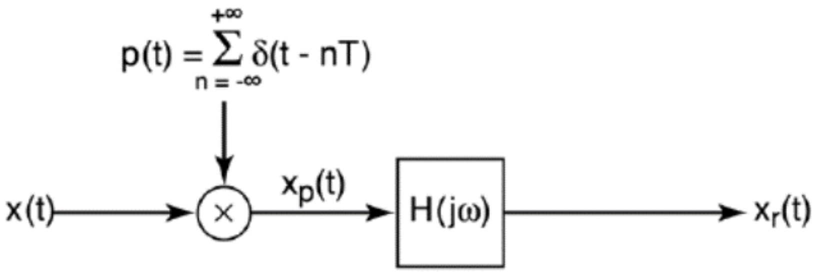
$X_p(j\omega)$ drawn assuming

$$\omega_s - \omega_M > \omega_M$$

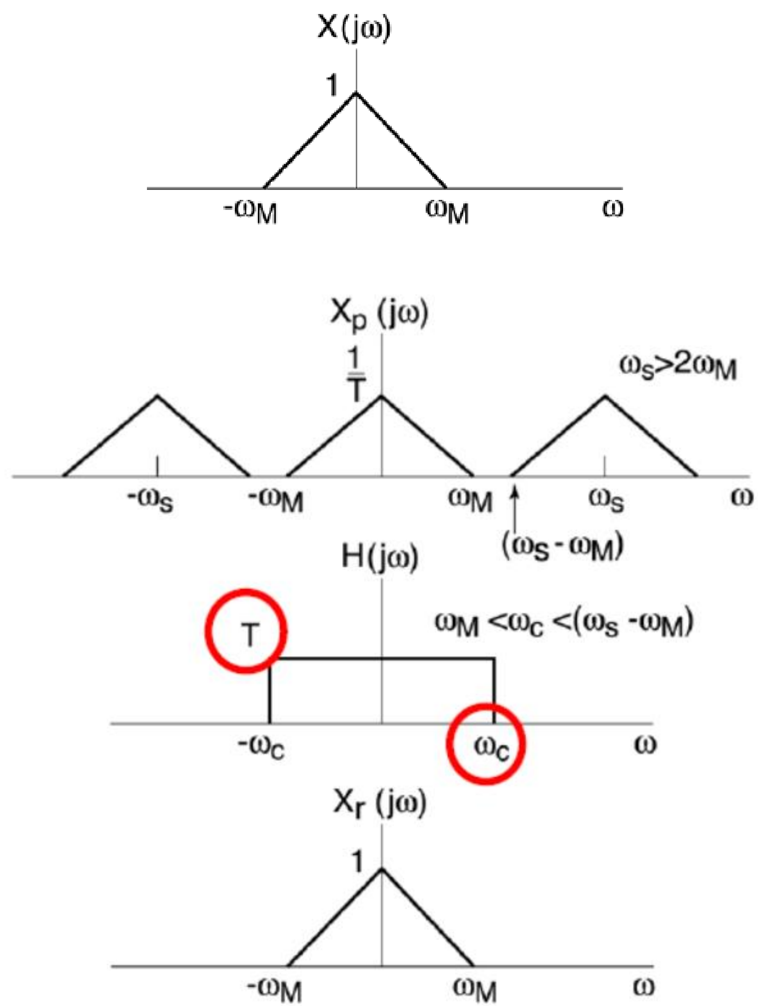
$$i.e. \quad \omega_s > 2\omega_M$$

No overlap between shifted spectra

- Reconstruction of x(t) from sampled signals



If there is no overlap between shifted spectra, a LPF can reproduce $x(t)$ from $x_p(t)$



Suppose $x(t)$ is band-limited, so that

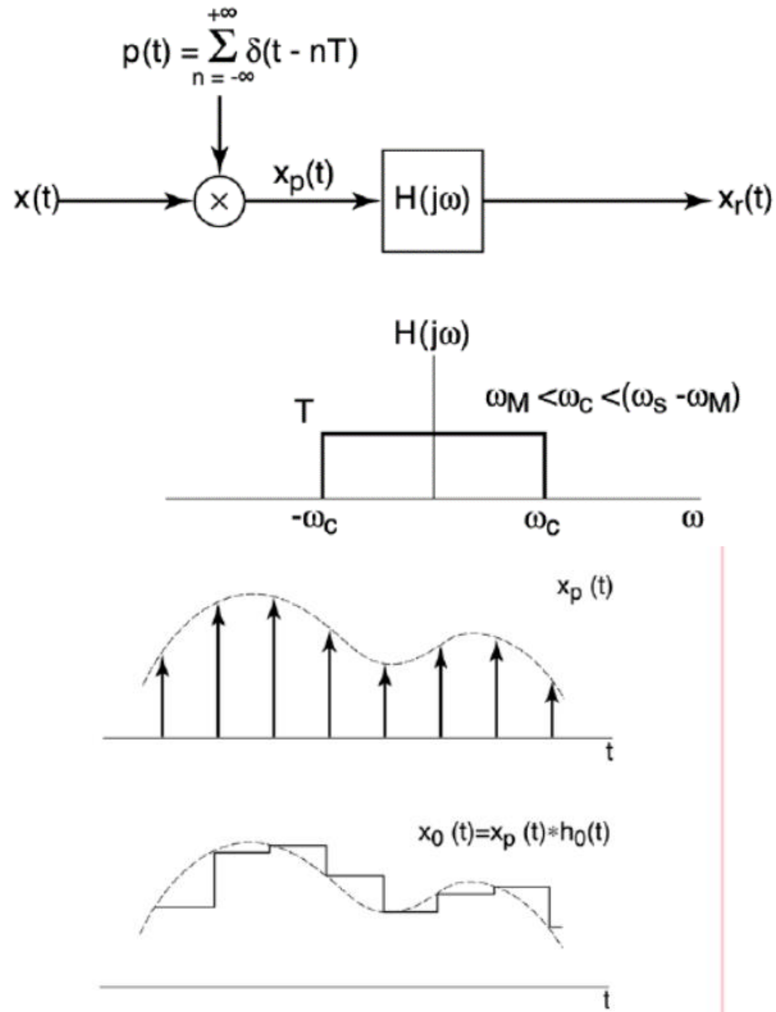
$X(j\omega)=0$ for $|\omega| > \omega_M$

Then $x(t)$ is uniquely determined by its

samples $\{x(nT)\}$ if

where $\omega_s = 2\pi/T$

- Observations
  - (1) In practice, we obviously don't sample with impulses or implement ideal lowpass filters — One practical example: The Zero-Order Hold
  - (2) Sampling is fundamentally a time varying operation, since we multiply x(t) with a time-varying function p(t). However, H(jω) is the identity system (which is TI) for band-limited x(t) satisfying the sampling theorem ($\omega_s > 2\omega_M$).
  - (3) What if $\omega_s <= 2\omega_M$? Something different: more later.



Sampling Theorem:

Let us be a band − limited signal with $X(j\omega)=0$ *for* $|\omega|>\omega_m$. *Then* $x(t)$ *is uniquely determined by its samples* $x(nT_s),\ n=0,\pm1,\pm2,\ldots,$ *if*

$$\omega_s > 2\omega_m$$

where $\omega_s = \dfrac{2\pi}{T_s}$

Given these samples, we can reconstruct $x(t)$ *by generating a periodic impulse train in which successive impulse have amplitudes that are successive sample values. This impulse train is then processed through an ideal lowpass filter with gain* T *and cutoff frequency* ω_c. *if*

$$\omega_s - \omega_m > \omega_c > \omega_m$$

the resulting output signal will exactly equal $x(t)$.

- Example:

Consider a band-limited signal $x(t)$ *with* $X(j\omega)=0$ *for* $|\omega|>\omega_m$.

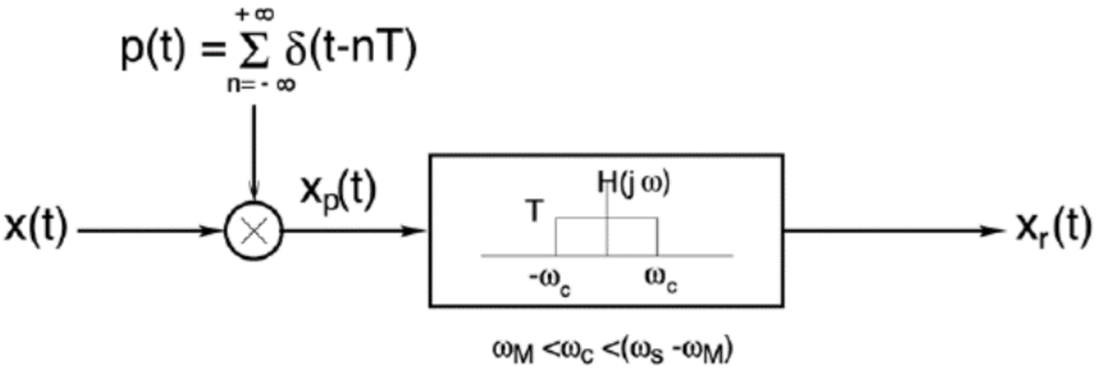
Determine the Nyquist rate for the following signals:

(1) $2x(t)+1$

(2) $x^2(t)$

(3) $\dfrac{dx(t)}{dt}$

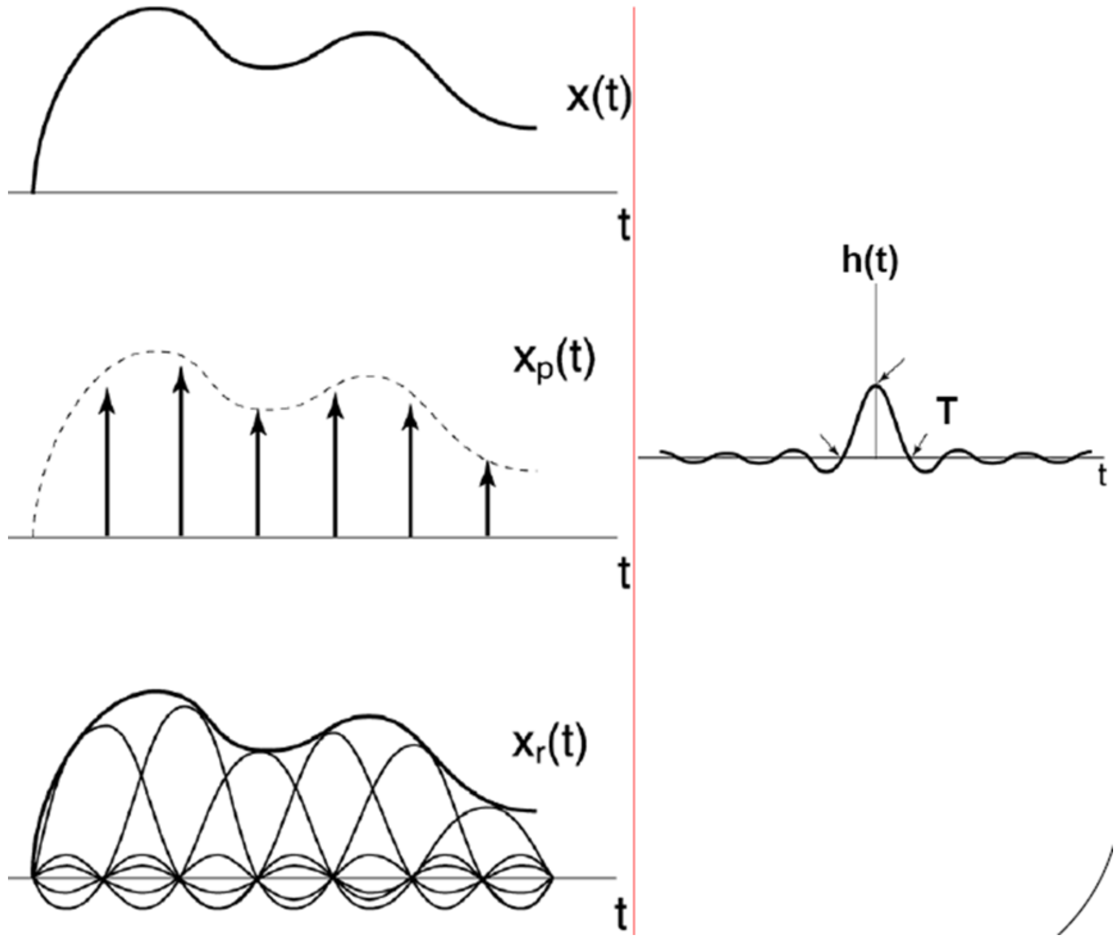
- Time-Domain Interpretation of Reconstruction of Sampled Signals —Band-Limited Interpolation



$$\begin{aligned} x_r(t) &= x_p(t) * h(t) \quad , \quad \text{where } h(t) = \frac{T \sin \omega_c t}{\pi t} \\ &= \left(\sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT) \right) * h(t) \\ &= \sum_{n=-\infty}^{\infty} x(nT)h(t-nT) = \sum_{n=-\infty}^{\infty} x(nT) \frac{T \sin[\omega_c(t-nT)]}{\pi(t-nT)} \end{aligned}$$

The lowpass filter interpolates the samples *assuming* $x(t)$ contains no energy at frequencies >= ω_c

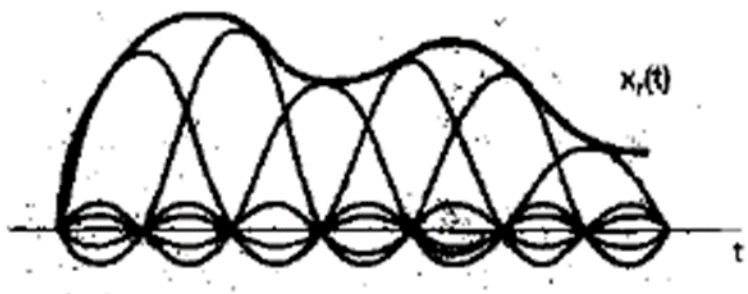
- Graphic Illustration of Time-Domain Interpolation
  - Original CT signal
  - After Sampling
  - After passing the LPF



- Interpolation Methods (1): Band-limited Interpolation: ideal LPF, i.e. sinc function in time domain

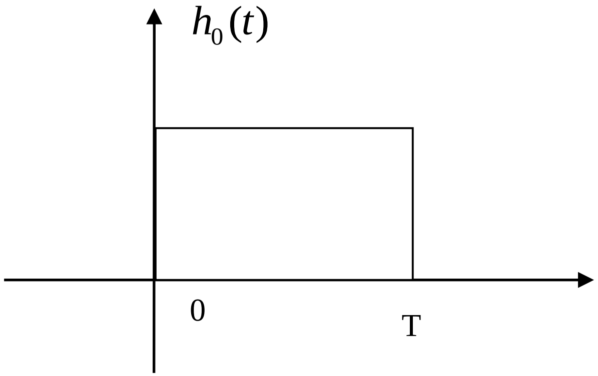
$$x_r(t) = \sum_n x(nT_s) \cdot \delta(t - nT_s) * \frac{\omega_c}{\pi} Sa(\omega_c \tau)$$

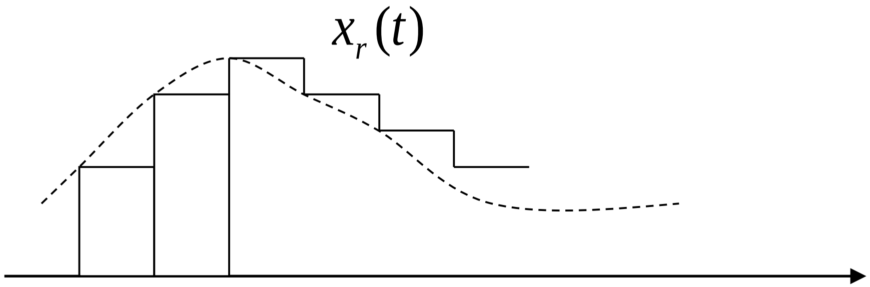
$$= \sum_n \frac{\omega_c}{\pi} x(nT_s) \cdot Sa[\omega_c (t - nT_s)]$$



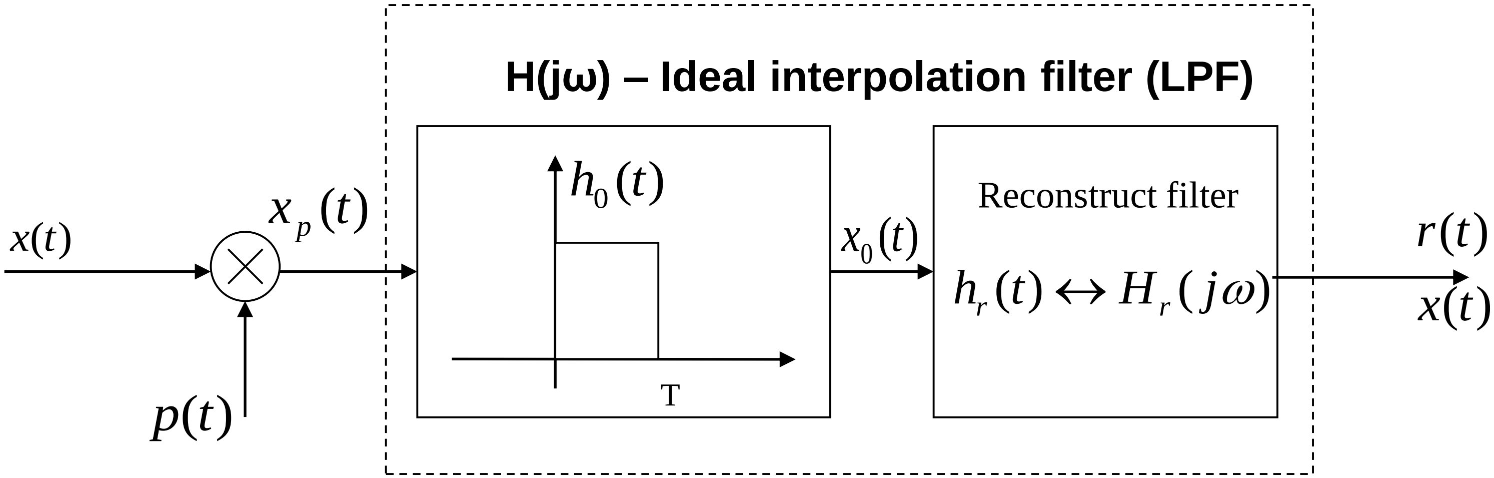
(c)

- Interpolation Methods (2): Zero-Order Hold



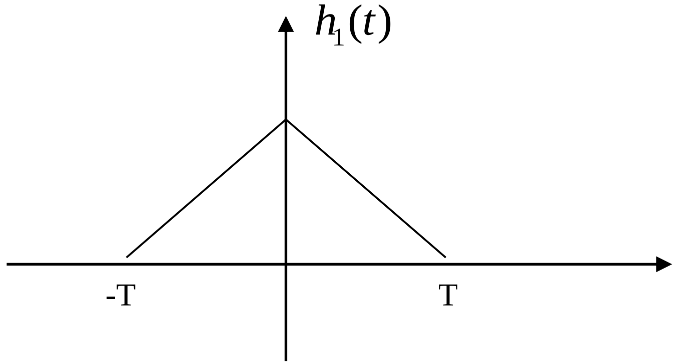


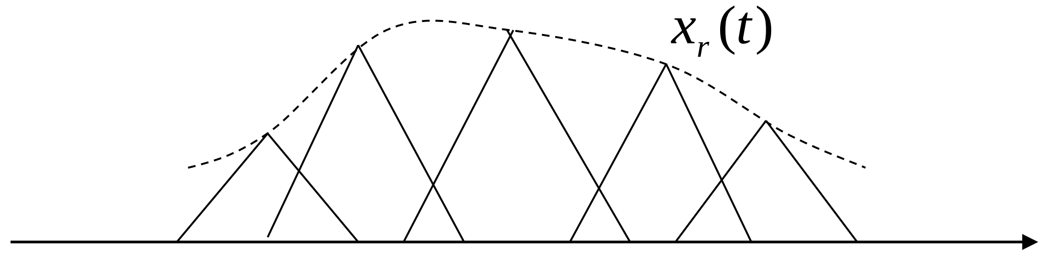
$$H_0(j\omega) = e^{-j\omega T/2}\left[\frac{2\sin(\omega T/2)}{\omega}\right]$$



$$H_r(j\omega) = \frac{H(j\omega)}{H_0(j\omega)} = \frac{e^{j\omega\frac{T}{2}} \cdot H(j\omega)}{\dfrac{2\sin(\omega T/2)}{\omega}}$$

- Interpolation Methods (3): First-Order Hold —Linear interpolation

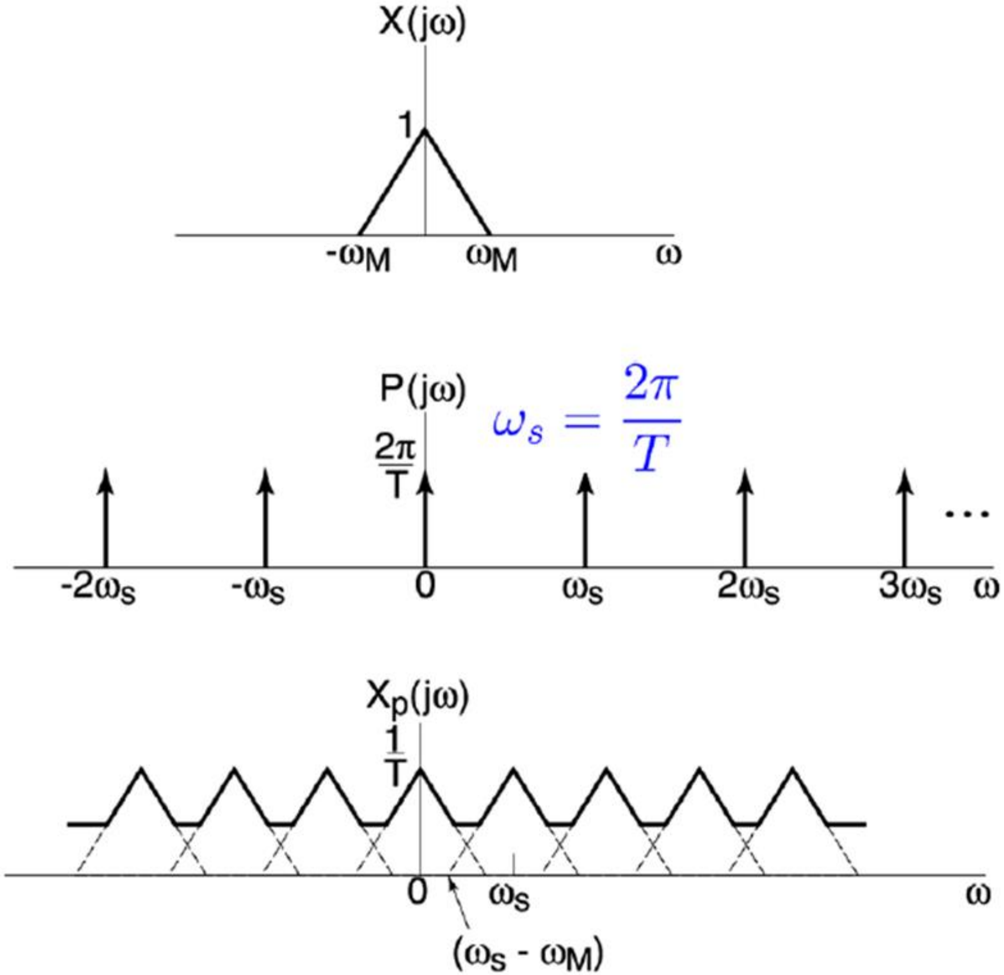


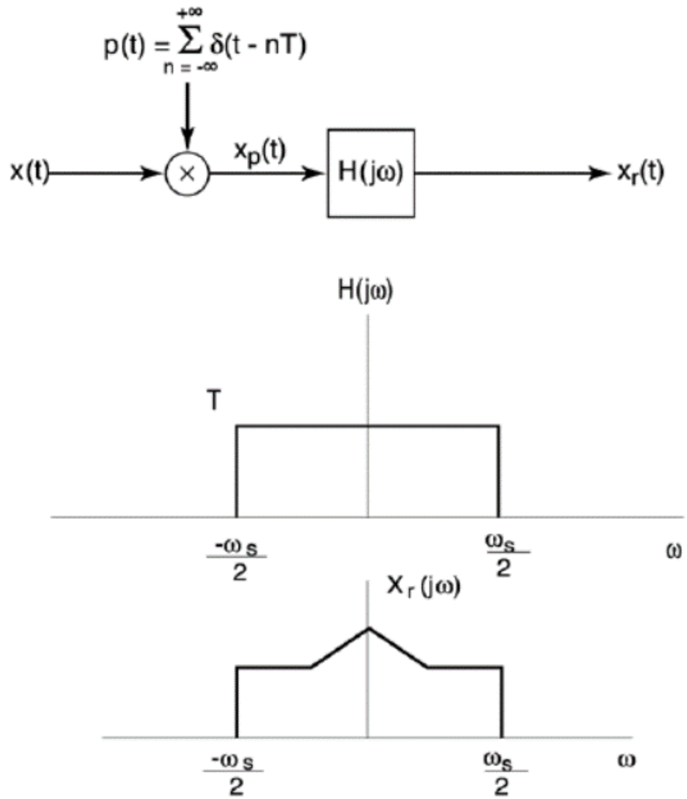


$$H_1(j\omega) = \frac{1}{T}\left[\frac{\sin(\omega T/2)}{\omega/2}\right]^2$$

- Under sampling and Aliasing

When $\omega_s \leqq 2\omega_M$ => Under-sampling





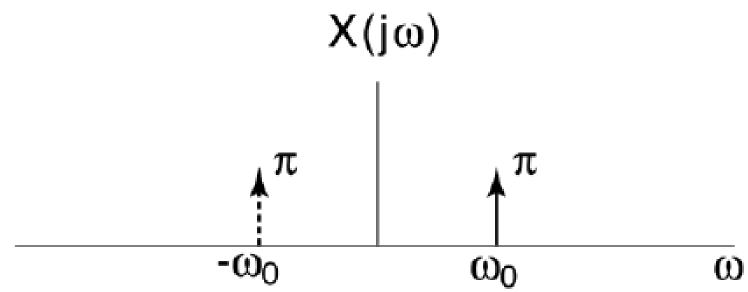
$X_r(j\omega) \neq X(j\omega)$

Distortion due to *aliasing*

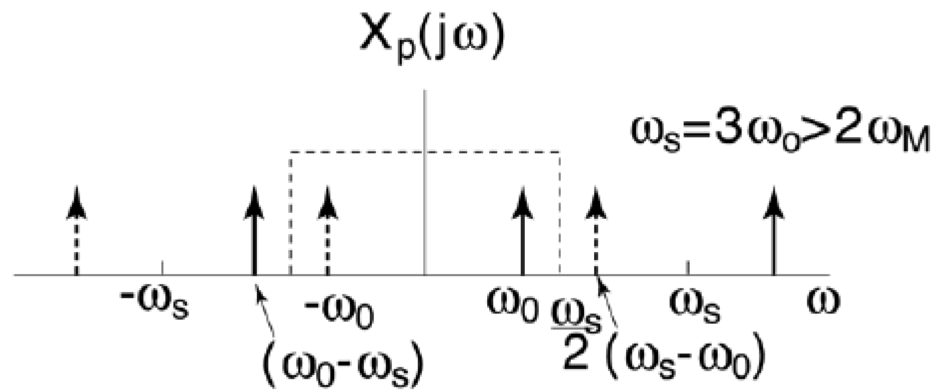
— Higher frequencies of $x(t)$ are "folded back" and take on the "aliases" of lower frequencies

— Note that at the sample times, $x_r(nT) = x(nT)$

- Example:



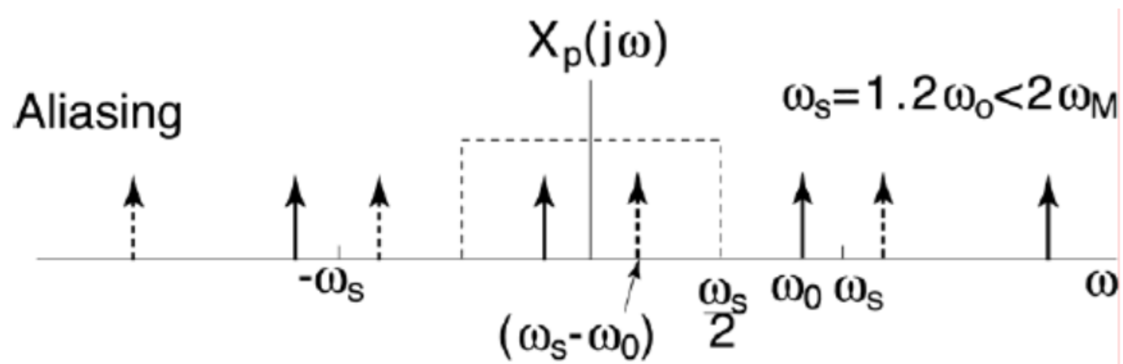
X(t) = cos(ω_ot + Φ)



Sampling of *cosω_ot*

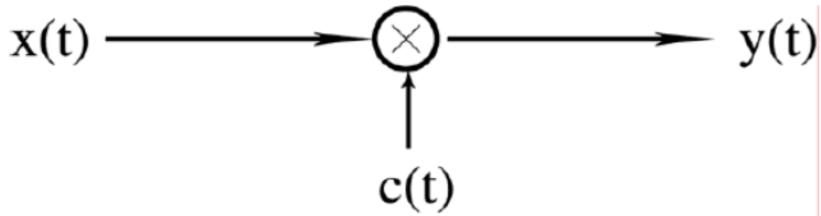
Aliasing case:

Then with the ideal LPF with cut off frequency of $\omega_M < \omega_c < \omega_s - \omega_o$, the reconstructed signal is cos((ω_s-ω_o)t)



Ref. Q7.38

- Example: AM with an Arbitrary Periodic Carrier



C(*t*) – periodic with period *T*, carrier frequency $\omega_c = 2\pi/T$

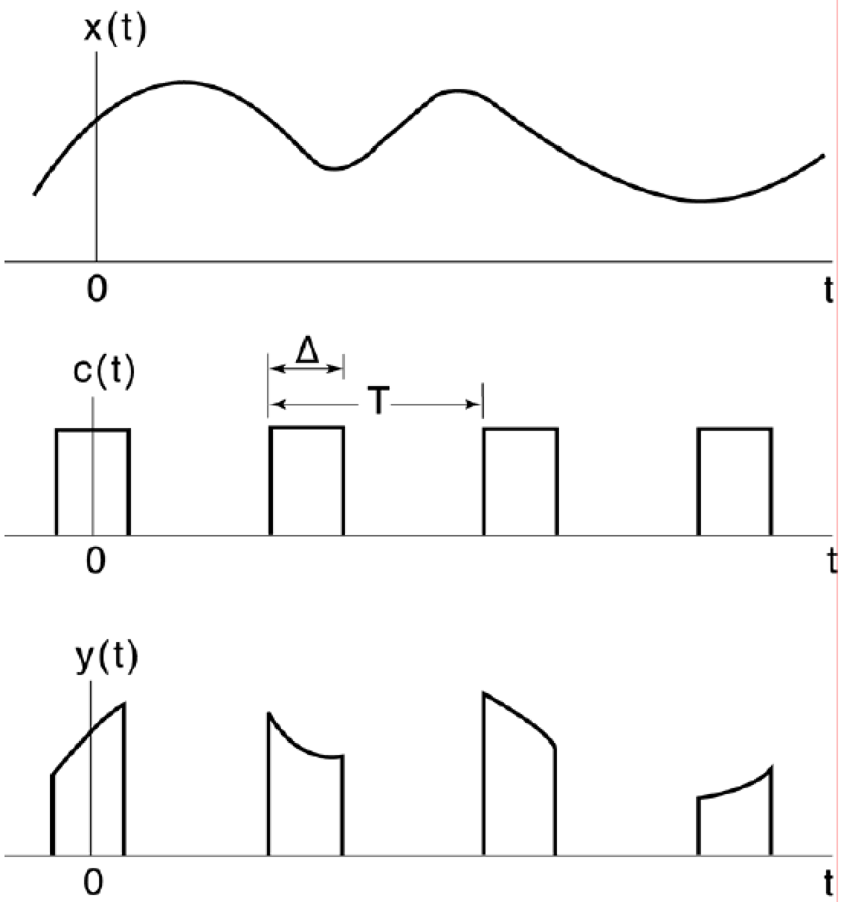
$$C(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_c) \quad \left(a_k = \frac{1}{T} \text{ for impulse train }\right)$$

$$\Downarrow$$

$$Y(j\omega) = \frac{1}{2\pi} X(j\omega) * C(j\omega) = X(j\omega) * \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_c)$$

$$= \sum_{k=-\infty}^{\infty} a_k X(j(\omega - k\omega_c))$$

- Example: Modulating a (Periodic) Rectangular Pulse Train



$$y(t) = x(t) \cdot c(t)$$

In practice, we can use a (periodic) rectangular pulse train instead of impulses, since the later is impractical

$$C(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_c)$$

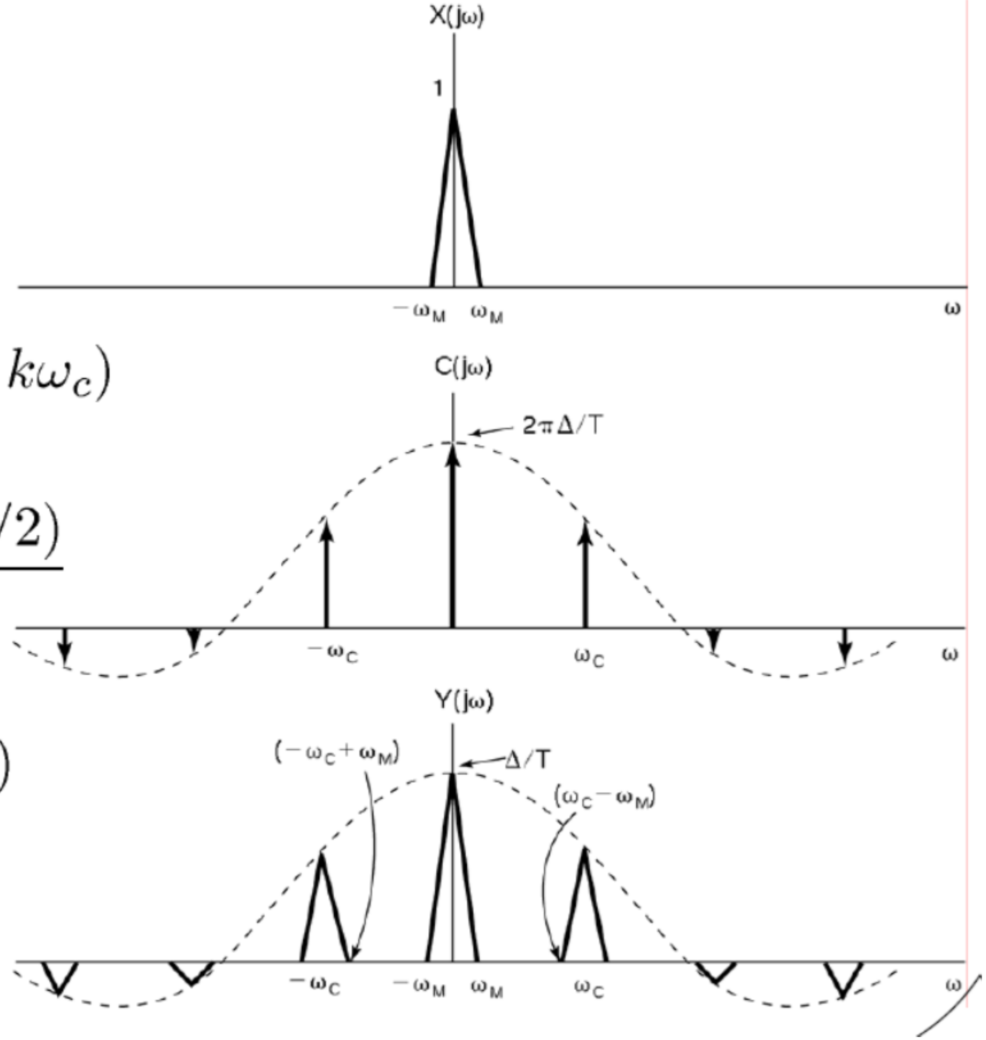
and

$$a_0 = \frac{\Delta}{T}, \quad a_k = \frac{\sin(k\omega_c\Delta/2)}{\pi k}$$

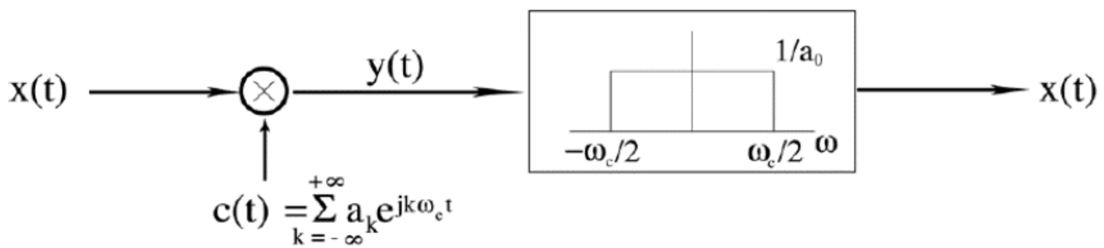
for rectangular pulse

$$Y(j\omega) = \frac{1}{2\pi} X(j\omega) * C(j\omega)$$

Drawn assuming:
$\omega_c > 2\omega_M$
Nyquist rate is met



- Discussions on modulating a (Periodic) Rectangular Pulse Train
  - 1) We get a similar picture with any c(t) that is periodic with period T
  - 2) As long as $\omega c = 2\pi/T > 2\omega M$, there is no overlap in the shifted and scaled replicas of $X(j\omega)$. Consequently, assuming $a0 \neq 0$:



x(t) can be recovered by passing y(t) through a LPF

  - 3) Pulse Train Modulation is the basis for Time-Division Multiplexing
  - Assign time slots instead of frequency slots to different channels, e.g. AT&T wireless phones
  - 4) Really only need samples{x(nT)} when $\omega c > 2\omega M \Rightarrow$ Pulse Amplitude Modulation

Topic

- 4.0 Introduction
- 4.1 The Continuous-Time Fourier Transform
- 4.2 The Fourier Transform for Periodic Signals
- 4.3 Properties of the Continuous-Time Fourier Transform
- 4.4 The Convolution Property
- 4.5 The multiplication Property
- 4.6 System Characterized by Linear Constant-Coefficient Differential Equations

LTI Systems Described by LCCDE's (Linear-constant-coefficient differential equations)

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Using the Differentiation Property

$$\frac{d^k x(t)}{dt^k} \longleftrightarrow (j\omega)^k X(j\omega)$$

Transform both sides of the

$$\sum_{k=0}^{N} a_k \cdot (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k \cdot (j\omega)^k X(j\omega)$$

$$\Downarrow$$

$$Y(j\omega) = \underbrace{\left[\frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}\right]}_{H(j\omega)} X(j\omega)$$

1) Rational, can use PFE to get $h(t)$
2) If $X(j\omega)$ is rational *e.g.* $x(t)=\Sigma c_i e^{-at} u(t)$ then $Y(j\omega)$ is also rational

PFE: Partial-fraction expansion

- Example：

$$\frac{d^2 y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

$$\Downarrow$$

$$H(j\omega) = \frac{j\omega + 2}{(j\omega)^2 + 4(j\omega) + 3}$$

- Zero-state response of LTI systems——Partial-fraction expansion method
- Example:

$$H(j\omega)=\frac{j\omega+2}{(j\omega)^2+4(j\omega)+3}=\frac{j\omega+2}{(j\omega+1)(j\omega+3)}=\frac{A_1}{j\omega+1}+\frac{A_2}{j\omega+3}$$

Let $j\omega=v$

then $A_1=(v+1)H(v)|_{v=-1}=\frac{v+2}{v+3}|_{v=-1}=\frac{1}{2}$

$A_2=(v+3)H(v)|_{v=-3}=\frac{v+2}{v+1}|_{v=-3}=\frac{1}{2}$

$$\therefore H(j\omega)=\frac{\frac{1}{2}}{j\omega+1}+\frac{\frac{1}{2}}{j\omega+3}$$

and $\frac{1}{j\omega+\alpha}\leftrightarrow e^{-\alpha t}u(t)$

$$\therefore h(t)=\frac{1}{2}e^{-t}u(t)+\frac{1}{2}e^{-3t}u(t)$$

- Example: $x(t)=e^{-t}u(t)$ To calculate the zero-state response of the system discussed in previous example

$$Y(j\omega)=H(j\omega)X(j\omega)=\frac{j\omega+2}{(j\omega+1)^2(j\omega+3)}=\frac{A_{11}}{j\omega+1}+\frac{A_{12}}{(j\omega+1)^2}+\frac{A_2}{j\omega+3}$$

$$A_{11}=\frac{1}{(2-1)!}\frac{d}{dv}[(v+1)^2Y(v)]|_{v=-1}=\frac{d}{dv}[\frac{v+2}{v+3}]|_{v=-1}=\frac{1}{(v+3)^2}|_{v=-1}=\frac{1}{4}$$

$$A_{12}=(v+1)^2Y(v)|_{v=-1}=\frac{1}{2}$$ high-order pole point

$$A_2=(v+3)Y(v)|_{v=-3}=\frac{v+2}{(v+1)^2}|_{v=-3}=-\frac{1}{4}$$

$$\therefore Y(j\omega)=\frac{\frac{1}{4}}{j\omega+1}+\frac{\frac{1}{2}}{(j\omega+1)^2}-\frac{\frac{1}{4}}{j\omega+3}$$

$$\because e^{-\alpha t}u(t)\leftrightarrow\frac{1}{\alpha+j\omega}$$

$$te^{-\alpha t}u(t)\leftrightarrow j\cdot\frac{d}{d\omega}[\frac{1}{\alpha+j\omega}]=\frac{1}{(\alpha+j\omega)^2}$$

$$\therefore y(t)=[\frac{1}{4}e^{-t}+\frac{1}{2}te^{-t}-\frac{1}{4}e^{-3t}]u(t)$$

- Homework
  - BASIC PROBLEMS WITH ANSWER: 4.1, 4.4
  - BASIC PROBLEMS: 4.21, 4.22, 4.25, 4.32, 6.21, 6.22, 7.3, 7.4, 8.22, 8.30

Q & A

Many Thanks